ON THE SCALING LIMITS OF GALTON WATSON PROCESSES IN VARYING ENVIRONMENT

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ABSTRACT. Renormalized sequences of Galton Watson processes converge to Continuous State Branching Processes (CSBP), characterized by a Lévy triplet of two numbers and a measure. This paper investigates the case of Galton Watson processes in varying environment and provides an explicit sufficient condition for finite-dimensional convergence in terms of convergence of a characteristic triplet of measures. We recover then classical results on the convergence of Galton Watson processes and we can add exceptional environments provoking positive or negative jumps at fixed times. We also apply this result to derive new results on the Feller diffusion in varying environment and branching processes in random environment. Our approach relies on the backward differential equation satisfied by the Laplace exponent and provides results about explosion, absorption and extinction. Thus, this paper exhibits a general class of CSBP in varying environment which is characterized by a triplet of measures. This provides a first step towards characterizing time-inhomogeneous, continuous-time and continuous state space processes which satisfy the branching property.

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1. INTRODUCTION

Since the pioneering work of Lamperti [27], it is known that continuous-state branching processes (CSBP) are the only possible scaling limits of Galton-Watson (GW) branching processes and that every CSBP can be realized in this way (see the following section for definitions). Another characterization of CSBP's was stated in Lamperti [26] who claimed that they are in one-to-one correspondence with spectrally positive Lévy processes killed upon reaching 0 via a random time-change called the Lamperti transformation, see Caballero et al. [11] for a discussion of various proofs of this fundamental result. Grimvall [19] established general necessary and sufficient conditions for a

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sequence of renormalized GW processes to converge. These conditions involve the asymptotic behavior of triangular arrays for which explicit necessary and sufficient conditions have been known for a long time, see for instance Gnedenko and Kolmogorov [16]. Finally, Ethier and Kurtz [14, Chapter 9] gave another proof of these results via time-change arguments. Hence to a large extent, the asymptotic behavior and the structure of the limiting processes of GW processes are well understood.

The present paper aims at extending this understanding to the case of GW processes in varying (and random) environment. Recently, there has been a considerable interest for GW processes in random environment, in particular about problems related to the survival behavior in the critical and subcritical regime (see, e.g., [1, 7, 15, 20]) or large deviations (see, e.g., [5, 9]). Similarly, branching diffusions in varying environment have attracted attention, in part for biological motivations, see among others [6, 8, 17] and references therein. Nonetheless, little seems to be known about the asymptotic behavior of GW processes in random or varying environment, except for the important special case of finite variance, see [8, 10, 24, 25].

The main result of the present paper establishes a sufficient condition for a sequence of GW processes in varying environment to converge in the sense of finite-dimensional distributions. The assumptions are stated in terms of the convergence of a characteristic triplet and exhibit an interesting class of processes in continuous time: CSBP in varying (and then random) environment which could be characterized by a triplet of measures. The drift term, equivalently described by a real-valued function, is here assumed to have local variations to get general convergence with tractable assumptions. Our result is thus applied to various examples, in particular to GW processes, Feller branching processes in varying environment and branching processes in random environment. Also, our approach, which relies on the convergence of Laplace transforms, provides qualitative properties on the limiting processes.

Scaling limits of GW processes. GW processes are classical Markov chains for population dynamics where individuals reproduce independently of each other and with the same reproduction law, see for instance Athreya and Ney [2]. Thus if Z_i denotes the size of the population at time $i \geq 0$, the process $(Z_i, i \geq 0)$ obeys the following recursion:

$$Z_{i+1} = N_{i,1} + \cdots + N_{i,Z_i}$$

where $(N_{i,k}, i, k \geq 0)$ are i.i.d. with common distribution the reproduction law. Another equivalent characterization of GW processes is through the *branching property*. Namely, GW processes are the only discrete-time, $\mathbb{N}$-valued Markov chains $(Z^j, j \geq 0)$, with Z^j the law of the Markov chain started at $Z_0^j = j$, such that Z^{j+k} for $j, k \geq 0$ is equal in distribution to $Z^j + \widetilde{Z}^k$ with $\widetilde{Z}^k$ a copy of Z^k independent of Z^j.

Scaling limits of GW processes have been studied since Lamperti [27]. In general there may be centering terms, but we will only be interested here in scaling limits obtained by starting a process from a large initial state, speeding up time and scaling in space. Typically, we consider a sequence $(Z^{(n)}, n \geq 1)$ of GW processes, where $Z^{(n)}$ has reproduction law μ_n and starts with $Z_0^{(n)} = n$ individuals, a sequence $(\vartheta_n, n \geq 1)$ of positive real numbers going to infinity, which will be called the *speed* of the GW process, and we consider the sequence $(X_n, n \geq 1)$ of rescaled processes defined by

$$X_n(t) = \frac{1}{n} Z^{(n)}_{\lfloor \vartheta_n t \rfloor}, \ t \geq 0.$$

The asymptotic behavior of (X_n) yields relevant approximations for phenomena such as evolution of species, with both large initial populations and long time scales. The most interesting case is when the sequence $(Z^{(n)}, n \geq 1)$ is near-critical, meaning that the mean of μ_n is closer and closer to one. Indeed, in the strictly super- and subcritical cases, the processes evolve rapidly (one does not need to speed up time) and, for our purposes, essentially deterministically. Grimvall [19] has proved that the finite-dimensional convergence of (X_n) is equivalent to the convergence in distribution of the sequence

$$\left(n^{-1} \sum_{i=1}^{\lfloor n\vartheta_n \rfloor} (N_{i,n} - 1),\ n \geq 1 \right),$$

where for each $n \geq 1$, $(N_{i,n}, i \geq 1)$ are i.i.d. with common distribution μ_n. Necessary and sufficient conditions for this convergence to hold are well-known, see the book by Gnedenko and Kolmogorov [16] on the convergence of infinitesimal triangular arrays. The infinitesimal assumption for triangular arrays corresponds precisely to the near-critical assumption for branching processes.

CSBP's are the continuous counterparts of GW processes, they are defined via a generalization of the branching property. Indeed, CSBP's are the only continuous-time, $[0,\infty]$-valued Markov processes $(X^x, x \geq 0)$, with X^x the law of the Markov process started at $X^x(0) = x$, such that X^{x+y} for $x, y \geq 0$ is equal in distribution to $X^x + \widetilde{X}^y$ with $\widetilde{X}^y$ a copy of X^y independent of X^x. As mentioned earlier, Lamperti [27] proved that if the above sequence (X_n) converges then the limit must be a CSBP, and that any CSBP can be approximated in this way.

Silverstein [28] gives a useful characterization of CSBP in terms of their Laplace transform. The branching property ensures that if X is a CSBP, then it satisfies

$$\mathbb{E}\left(\exp(-\lambda X(t)) \mid X(0) = x\right) = \exp(-x u(t,\lambda))$$

for some function $u(t,\lambda)$ called the Laplace exponent. Silverstein [28] has proved that for each $\lambda > 0$, the function $u(\cdot,\lambda)$ is characterized by the following differential equation:

$$u(t,\lambda) = \lambda + \int_0^t \psi(u(x,\lambda))\, dx,\ \ t \geq 0,$$

where

$$\psi(\lambda) = \alpha\lambda - \beta\lambda^2 + \int_0^\infty \left(1 - e^{-\lambda x} - \frac{\lambda x}{1+x^2}\right) \nu(dx)$$

for some $\alpha \in \mathbb{R}$, $\beta \geq 0$ and ν a measure on $(0,\infty)$ such that $\int_0^\infty x^2/(1+x^2)\nu(dx)$ is finite. The function ψ is called the *branching mechanism* of the CSBP, and we see in particular that a CSBP is characterized by a triplet (α, β, ν). This fact can also be seen from the Lamperti transformation which makes a one-to-one correspondence between CSBP's and spectrally positive Lévy process killed upon reaching 0, see Lamperti [26] and Caballero et al. [11].

Renormalization of GW processes in varying environment. In this paper, we want to extend some of the above results to the case of GW processes in varying environment. In the case of random environment this corresponds to adopting a quenched approach. In terms of evolution or population dynamics, it can be motivated both by slowly fluctuating conditions for the population and major catastrophes. The last one will correspond to non-critical environments. So now, for each $n \geq 1$ the process $Z^{(n)}$ is a GW process in

varying environment $(\mu_{i,n}, i \geq 1)$ with $\mu_{i,n}$ the reproduction law in the ith generation. Hence for $i \geq 0$ we have the recursion

$$Z_{i+1}^{(n)} = N_{i,1}^{(n)} + \cdots + N_{i,Z_i^{(n)}}^{(n)}$$

where the random variables $(N_{i,k}^{(n)}, i, k \geq 0)$ are independent and $N_{i,k}^{(n)}$ has distribution $\mu_{i,n}$. By analogy with the renormalization of GW processes, a natural way to renormalize the sequence $(Z^{(n)}, n \geq 1)$ is by considering $Z_0^{(n)} = n$, an onto and increasing function $\gamma_n : [0, \infty) \to \mathbb{N}$ and by defining the process X_n

$$X_n(t) = \frac{1}{n} Z_{\gamma_n(t)}^{(n)}, \ t \geq 0.$$

In the GW case we had $\gamma_n(t) = \lfloor \vartheta_n t \rfloor$ for some sequence (ϑ_n), but in general we need to consider more general functions γ_n. Indeed, in varying environment consider the case where for each $n \geq 1$ the environments are first all equal to some reproduction law μ_n^1 and then take the constant value μ_n^2. If $(\mu_n^i, n \geq 1)$ for each $i = 1, 2$ corresponds to a sequence of GW processes with speed (ϑ_n^i), then it is natural to take the function γ_n equal to the integer part of a piecewise linear function, which first takes slope ϑ_n^1 and then ϑ_n^2. Then (X_n) would converge to a process X which can informally be described as a "piecewise CSBP", i.e., X would be a CSBP with some branching mechanism ψ^1 for some time after which it would behave like a CSBP with some other branching mechanism ψ^2.

Related results. The asymptotic behavior of GW processes in varying environment has been thoroughly studied in the finite variance case, see for instance Keiding [24], Kurtz [25] or Borovkov [10]. One of the simplification in the finite variance case is that the speeds of finite variance GW processes are all the same, and equal to n (to be more precise, the time and space scales need to be the same, and equal to $1/(1-\rho_n)$ with ρ_n the mean of the offspring distribution). In particular, there is a natural way to speed up those GW processes, namely by considering the natural choice $\gamma_n(t) = \lfloor nt \rfloor$, which turns out to be a good candidate.

In the finite variance case, X_n converges to a branching diffusion (also called Feller diffusion) in varying environment, which may have positive or negative jumps at a fixed time. Getting a general extinction criterion in this case is a challenging problem. In contrast with the case of constant environment, the average behavior of the process, given by the drift part, does not lead to the good criterion because of possible important variations. We refer to Section 3.2 for the explicit criterion in the Feller case.

Moreover, in random environment, we can observe different speeds of extinction in the subcritical case. This phenomenon is well-known in the discrete case, see, e.g., [15, 20]. We refer to [8] for first results in the continuous framework, more precisely for branching Feller diffusion in random environment.

Besides time-inhomogeneous branching diffusions, more general time-inhomogeneous branching processes appear in the related literature on superprocesses. Dynkin [12] has built superprocesses whose mass X, which satisfies the branching property, obeys through their Laplace exponent $u(s, t, \lambda) = -\log \mathbb{E}(e^{-\lambda X(t)} \mid X(s) = 1)$ to the equation

$$u(s, t, \lambda) = \lambda + \int_s^t \psi(x, u(x, t, \lambda)) K(dx),$$

where K is some σ-finite measure and $\psi(t, \lambda)$ is a time-varying branching mechanism, i.e., for each $t \geq 0$ the function $\psi(t, \cdot)$ is a branching mechanism with characteristics

$(\alpha_t, \beta_t, \nu_t)$. These processes do not allow for explosion, but this was allowed by El-Karoui and Roelly [13] via martingale method when $K(dt) = dt$. We can say that these processes are characterized by a triplet of measures $(\alpha_t K(dt), \beta_t K(dt), \nu_t K(dt))$ which are in some sense all absolutely continuous with respect to one another, since they are all absolutely continuous with respect to K. The processes that we consider are slightly more general, since we will indeed characterize our limiting objects by a triplet of measures, but that need not be absolutely continuous with respect to one another.

In the time-homogeneous setting, K is Lebesgue measure. The absolutely continuous component part of K represents the infinitesimal evolutions while its singular part represents times of catastrophes, corresponding to non-critical environment: the mass makes a sudden jump. Jumps at a fixed time may occur when the measure K has an atom. Note that Dynkin [12] builds superprocesses starting from continuous-time, discrete state-space branching systems. Starting from continuous-time processes allows to get rid of many technical difficulties that we have to deal with. But our focus is different, since we want to understand the asymptotic behavior of GW processes in varying environment.

Organization of the paper. In Section 2 we set up the framework and notation and state the main results. Theorem 2.1 gives a sufficient condition for convergence in the sense of finite-dimensional distributions, while Corollaries 2.4 and 2.5 give criteria for almost sure absorption or explosion. Before proving these results, we examine in Section 3 their relevance. We first compare the criterion obtained to known optimal conditions obtained by Grimvall [19] in the time-homogeneous case. We then specify the convergence of GW in varying environment with bounded variance. We finally look at the case of branching processes in random environment. In Section 4.1 we give an outline of the proof of Theorem 2.1 and an intuitive explanation of the dynamics satisfied by our limiting processes. The rest of Section 4 is then devoted to the proof of Theorem 2.1, while Corollaries 2.4 and 2.5 are proved in Section 5.

2. Notation and main results

2.1. **General notation.** For each $n \geq 1$, we consider a Galton-Watson process in varying environment $(Z_i^{(n)}, i \geq 0)$. We fix the space scale equal to n while the time scale is allowed to vary over time. For $n \geq 1$, we consider a non-decreasing, càdlàg and onto function $\gamma_n : [0, \infty) \to \mathbb{N}$ (here and elsewhere, $\mathbb{N} = \{0, 1, \dots\}$ denotes the set of non-negative integers). We then define the renormalized process $(X_n(t), t \geq 0)$ via the following formula:

$$X_n(t) = \frac{1}{n} Z^{(n)}_{\gamma_n(t)}, \ t \geq 0.$$

Since $Z^{(n)}$ is a branching process, for each $\lambda \geq 0$ and $z, i, j \geq 0$ with $i \leq j$, one can write

$$\mathbb{E}\left[\exp\left(-\lambda Z_j^{(n)}\right) \mid Z_i^{(n)} = z\right] = \exp(-z v_n(i, j, \lambda))$$

for some function v_n. Then one can check that for any $\lambda, x, s, t \geq 0$ with $s \leq t$,

$$\mathbb{E}\left[\exp\left(-\lambda X_n(t)\right) \mid X_n(s) = x\right] = \exp(-x u_n(s, t, \lambda))$$

where $u_n(s, t, \lambda) := n v_n(\gamma_n(s), \gamma_n(t), \lambda / n)$. The Markov property implies the following composition rule: for any $0 \leq t_1 \leq t_2 \leq t_3$ and $\lambda \geq 0$,

$$u_n(t_1, t_3, \lambda) = u_n(t_1, t_2, u_n(t_2, t_3, \lambda)). \tag{1}$$

For $i \geq 0$ and $n \geq 1$ we note $t_i^n = \inf\{t \geq 0 : \gamma_n(t) = i\}$, so that $\gamma_n(t_i^n) = i$ by right-continuity of γ_n, $\mu_{i,n}$ the offspring distribution of generation i in $Z^{(n)}$, $\nu_{i,n}$ the measure on $\mathbb{R}$ with support included in $[-1/n, \infty)$ defined by

$$\nu_{i,n}[a,b] := n\mu_{i,n}[na+1, nb+1], \quad -1/n \leq a \leq b,$$

and $\alpha_{i,n}$, $\beta_{i,n}$ the two following (finite) real numbers:

$$\alpha_{i,n} := \int_{[-1/n,\infty)} \frac{x}{1+x^2} \nu_{i,n}(dx) \text{ and } \beta_{i,n} := \int_{[-1/n,\infty)} \frac{x^2}{2(1+x^2)} \nu_{i,n}(dx)$$

which can be rewritten in terms of the $(\mu_{i,n})$ as follows:

$$\alpha_{i,n} = \sum_{k=-1}^{\infty} \frac{k}{1+(k/n)^2} \mu_{i,n}\{k+1\} \text{ and } \beta_{i,n} = \frac{1}{2n} \sum_{k=-1}^{\infty} \frac{k^2}{1+(k/n)^2} \mu_{i,n}\{k+1\}.$$

From now on let $\mathscr{B}$ denote the Borel subsets of $\mathbb{R}$. For $n \geq 1$, let α_n and β_n be the measures on $\mathbb{R}$ with support included in $(0, \infty)$ defined by

$$\alpha_n(A) := \sum_{i \geq 1} \mathbb{1}_{\{t_i^n \in A\}} \alpha_{i-1,n} \text{ and } \beta_n(A) := \sum_{i \geq 1} \mathbb{1}_{\{t_i^n \in A\}} \beta_{i-1,n}, \; A \in \mathscr{B},$$

and let ν_n be the measure on $\mathbb{R}^2$ with support included in $[-1/n, \infty) \times (0, \infty)$ defined by

$$\nu_n(A \times B) := \sum_{i \geq 1} \mathbb{1}_{\{t_i^n \in B\}} \nu_{i-1,n}(A), \; A, B \in \mathscr{B}.$$

Then the integral $\nu_n(f)$ of a positive function $f : \mathbb{R}^2 \to [0, \infty)$ is given by

$$\nu_n(f) = \sum_{i \geq 1} \int f(x, t_i^n) \nu_{i-1,n}(dx).$$

From now on we identify any signed measure α with its corresponding càdlàg function of locally finite variation, see for instance Chapter 3 in Kallenberg [23], so we note indifferently $\alpha((s,t])$, $\alpha(s,t]$ or $\alpha(t) - \alpha(s)$ for $0 \leq s \leq t$. In particular, since $\alpha_n\{0\} = \beta_n\{0\} = 0$ and $t_i^n \Leftrightarrow i \leq \gamma_n(t)$ one can rewrite

$$\alpha_n(t) = \alpha_n(0,t] = \sum_{i=0}^{\gamma_n(t)-1} \alpha_{i,n} \text{ and } \beta_n(t) = \beta_n(0,t] = \sum_{i=0}^{\gamma_n(t)-1} \beta_{i,n}, \; t \geq 0,$$

where from now on we adopt the convention $\sum_a^b = 0$ if $b < a$. We write $|\alpha|$ for the total variation of α, and in particular it holds that $\left|\int f d\alpha\right| \leq \int |f| d|\alpha|$ for any measurable function f. Note that one has $|\alpha_n|(A) = \sum_{i \geq 1} \mathbb{1}_{\{t_i^n \in A\}} |\alpha_{i-1,n}|$.

2.2. **Main result.**

The main result of the paper, Theorem 2.1 below, relates the asymptotic behavior of the sequence (X_n) in the sense of finite-dimensional distributions to the asymptotic behavior of the triplet $(\alpha_n, \beta_n, \nu_n)$. Typically, we aim at controlling Laplace transforms of the kind

$$\mathbb{E}\left(\exp(-\lambda_1 X_n(t_1) - \cdots - \lambda_k X_n(t_k)) \;\middle|\; X_n(0) = 1\right)$$

with $\lambda_i, t_i \geq 0$. Because of the Markov and branching properties, this boils down to study the convergence of $\mathbb{E}(e^{-\lambda X_n(t)} \mid X_n(s) = x)$ with $s \leq t$ (see the proof of Corollary 2.3). This latter is equivalent to the convergence of the Laplace exponent u_n.

So we need to study the process X_n between time s and t. In general, we may run into complications if in this time-interval there is a *bottleneck* which sends the process to 0. Indeed, remember that we are considering GW processes in varying environment, and so even if most offspring distributions are well-behaved (near-critical) nothing prevents

a catastrophic environment to occur from time to time. This is in sharp contrast with standard GW processes, where all offspring distributions are near-critical.

Such a bottleneck can potentially create a problem of indetermination. Because CSBP's may not be conservative, i.e., they may explode in finite time. Then an indetermination of the kind $\infty \times 0$ can arise if our time-inhomogeneous process first explodes and then goes through a bottleneck. This indetermination is especially difficult to interpret since the pre-limit GW processes cannot explode in finite time. In Theorem 2.1, we first focus on the case where between time s and t the process does not go through any such bottleneck. The remaining cases are analyzed in Corollaries 2.4 and 2.5.

To formalize the above idea, we introduce for $t \geq 0$ the following time $\wp(t)$:

$$\wp(t) := \sup\left\{ s \leq t : \lim_{\varepsilon \to 0} \liminf_{n\to\infty} \inf_{v \in [s,t]} \mathbb{P}\left(X_n(t) > \varepsilon \,\middle|\, X_n(v) = 1 \right) = 0 \right\} \tag{2}$$

with the convention $\sup \varnothing = 0$. Intuitively, $\wp(t)$ is the time of the last bottleneck before time t. Hence by definition, for $s \in (\wp(t), t]$ there is no bottleneck between time s and t. It prevents the process from being absorbed a.s. and enables us to study the asymptotic behavior of $u_n(s,t,\lambda)$.

Theorem 2.1 (Behavior on $[\wp(t), t]$)**.** *Let α be a càdlàg function of locally finite variation, β be an increasing càdlàg function and ν be a measure on $\mathbb{R}^2$ with support included on $(0,\infty) \times (0,\infty)$ such that $\nu((0,\infty) \times (0,t]) < +\infty$ for every $t > 0$. Assume that*

(A1) $\alpha_n(t) \to \alpha(t)$, $|\alpha_n|(t) \to |\alpha|(t)$, $\beta_n(t) \to \beta(t)$ *and* $\nu_n([x,\infty) \times (0,t]) \to \nu([x,\infty) \times (0,t])$

as n goes to infinity, for every $t \geq 0$ and every $x > 0$ such that $\nu(\{x\} \times (0,t]) = 0$. Assume moreover that

(A2) $\alpha_{\gamma_n(t),n} \to \alpha\{t\}$, $\beta_{\gamma_n(t),n} \to \beta\{t\}$ *and* $\nu_{\gamma_n(t),n}[x,\infty) \to \nu([x,\infty) \times \{t\})$

as n goes to infinity, for every t such that either $\alpha\{t\} \neq 0$, $\beta\{t\} \neq 0$ or $\nu((0,\infty) \times \{t\}) \neq 0$ and x such that $\nu(\{x\} \times \{t\}) = 0$.

Then for every $t, \lambda > 0$ and $s \in [\wp(t), t]$, there exists $u(s,t,\lambda)$ such that

$$\lim_{n\to+\infty} u_n(s,t,\lambda) = u(s,t,\lambda).$$

Moreover, for any $t \geq 0$, $\alpha\{t\} \geq -1$, $\int_{(0,\infty)\times(0,t]} \left(1 \wedge x^2\right) \nu(dx\,dy)$ is finite and $\widetilde{\beta}\{t\} = 0$ where

$$\widetilde{\beta}(t) = \beta(t) - \int_{(0,\infty)\times(0,t]} \frac{x^2}{2(1+x^2)} \nu(dx\,dy).$$

Finally, for all fixed $t, \lambda > 0$, the function $s \in [\wp(t), t] \mapsto u(s,t,\lambda)$ is the unique càdlàg solution of the backwards differential equation

$$\begin{aligned} u(s,t,\lambda) = \lambda &+ \int_{(s,t]} u(y,t,\lambda)\alpha(dy) - \int_{(s,t]} (u(y,t,\lambda))^2 \widetilde{\beta}(dy) \\ &+ \int_{(0,\infty)\times(s,t]} \left(1 - e^{-xu(y,t,\lambda)} - \frac{xu(y,t,\lambda)}{1+x^2} \right) \nu(dx\,dy) \end{aligned} \tag{3}$$

such that $\inf_{v\in[s,t]} u(v,s,\lambda) > 0$ *for every* $\wp(t) < s \leq t$.

Remark 2.2. The assumption (A2) can be relaxed by replacing $\gamma_n(t)$ by an integer $i_n(t)$ such that $t^n_{i_n(t)} \leq t$ and $t^n_{i_n(t)} \to t$. Then only the proof of Lemma 4.8 needs minor modifications.

The assumption (A1) on the finiteness and convergence of $|\alpha_n|$ is used several times in the proof, in particular to make the solution of the backward differential equation converge via Lipschitz properties. Another approach [8, 25] allows to deal with infinite variations for α, but as far as we know, it is restricted to the finite variance framework and drift functions with infinite variation. Theorem 2.1 can be extended to get the convergence of the finite-dimensional distributions.

Corollary 2.3. *Let $x, t \geq 0$, $s \in [\wp(t), t]$, $I \geq 1$, $s \leq t_1 \leq \cdots \leq t_I \leq t$ and $\lambda_i > 0$ for $i = 1, \ldots, I$. Then under the notation and assumptions of Theorem 2.1, we have*

$$\lim_{n\to+\infty} \mathbb{E}\left(\exp\left(-\lambda_1 X_n(t_1) - \cdots - \lambda_I X_n(t_I)\right) \mid X_n(s) = x\right)$$
$$= \exp(-xu(s, t_1, \lambda_1 + u(t_1, t_2, \lambda_2 + u(\cdots, u(t_{I-1}, t_I, \lambda_I)\cdots)))).$$

2.3. **Behavior on** $[0, \wp(t)]$**.** Theorem 2.1 describes the asymptotic behavior of X_n on $[\wp(t), t]$. As discussed before this theorem, if $s < \wp(t)$ then between time s and t the process goes through at least one bottleneck that potentially sends it to 0, which may cause an indetermination of the kind $\infty \times 0$. To avoid this problem, we treat two special cases of interest.

- **Non-absorbing case:** there is no bottleneck, so that $\wp(t) = 0$ and Theorem 2.1 provides a picture on $[0, t]$.
- **Non-explosive case:** the process cannot explode, so that it is absorbed at 0 if it goes through a bottleneck and $u(s, t, \lambda) = 0$ if $s \leq \wp(t)$.

Corollary 2.4 provides a sufficient condition to be in the non-absorbing case, intuitively it should be enough that the average of each offspring distribution is bounded away from 0. On the other hand, Corollary 2.5 provides two sufficient conditions to be in the non-explosive case, one comes easily in terms of tightness of a suitable family of random variables and one, more demanding but more explicit, in terms of boundedness of first moments.

We emphasize that the following result holds with significantly weaker conditions than the conditions (A1) and (A2) needed for Theorem 2.1.

Corollary 2.4 (Non-absorbing case)**.** *Let $t > 0$. If*

$$\lim_{\varepsilon\to 0} \inf_{n\geq 1, s\in[0,t]} \mathbb{P}(X_n(t) \geq \varepsilon \mid X_n(s) = 1) > 0, \tag{4}$$

then $\wp(t) = 0$. Moreover, for (4) to hold it is enough that the two following conditions hold:

$$\sup_{n\geq 1} \left(|\alpha_n|(t) + \beta_n(t) + n^{-2} \sum_{i=0}^{\gamma_n(t)} \mu_{i,n}\{0\} \right) < +\infty$$

and for every $a > 0$,

$$\liminf_{n\to+\infty} \left(\inf_{0\leq i\leq \gamma_n(t)} \sum_{k=0}^{an} k\mu_{i,n}\{k\} \right) > 0.$$

Roughly speaking, the second assumption ensures that $\mu_{i,n}$ is not too close to δ_0. It avoids the almost sure absorption in one generation. Note that the condition on the sequence $(n^{-2} \sum_{i=0}^{\gamma_n(t)} \mu_{i,n}\{0\})$ is satisfied as soon as $(n^{-2}\gamma_n(t))$ is bounded. This is always the case in the constant environment case, where the fastest speed $\gamma_n(t) = \lfloor nt \rfloor$ is given by the finite variance case. This property seems to hold more generally, and it holds in the examples we study in Section 3.

We turn now to the problem of explosion. We know from the GW case that explosion may occur at a random time and we refer to Grey [18] for necessary and sufficient conditions. We specify a sufficient condition that guarantees that explosion almost surely does not occur; it is related to a first moment condition, which is also common in the GW case. Then Theorem 2.1 can be extended to the time interval $[0,t]$.

Corollary 2.5 (Non-explosive case)**.** *Fix $\lambda, t > 0$ and assume that the assumptions* (A1) *and* (A2) *of Theorem 2.1 hold. If*

$$\lim_{A\to\infty} \sup_{n\geq 1, v\in[s,t]} \mathbb{P}(X_n(v) \geq A \mid X_n(s) = 1) = 0, \tag{5}$$

for all $s \leq t$, then $\liminf_{n\to\infty} u_n(s,t,\lambda) = 0$ *for every $s < \wp(t)$. Moreover for* (5) *to hold, it is enough that*

$$\sup_{n\geq 1} \left(\int_{[-1/n,\infty)\times(0,t]} |x| \nu_n(dx\,dy) \right) < +\infty.$$

2.4. **Assumptions** (A1) **and** (A2)**, triangular arrays and processes with independent increments.** The assumptions (A1) and (A2) are reminiscent of conditions for the convergence of non-infinitesimal triangular arrays, see for instance Theorem VII.4.4 in Jacod and Shiryaev [22]. Relationships between the convergence of GW processes, triangular arrays and Lévy processes are well-known.

Grimvall [19] established general necessary and sufficient conditions for the convergence of GW processes in terms of some triangular arrays of rowwise i.i.d. random variables, see the introduction. Moreover, Jacod and Shiryaev [22] investigated the relationship between convergence of triangular arrays and the convergence of processes with independent increments. To a large extent, the two are are equivalent. Thus combining these results in the time-homogeneous case, we see that the convergence of a sequence of rescaled GW processes is equivalent to the convergence of corresponding Lévy processes. But this result can actually be directly obtained via time-change arguments, see, e.g., Helland [21] or Ethier and Kurtz [14, Chapter 9].

Our conditions (A1) and (A2) suggest that triangular arrays could play a role for the convergence of GW processes in varying environment; in view of Jacod and Shiryaev [22] this suggests in turn that processes with independent increments could also be interesting objects to consider. If this intuition turns out to be true, the time-homogeneous case suggests that the most efficient way to link GW processes in varying environment to processes with independent increments would be via time-change arguments. Nonetheless, it does not seem straightforward to extend the Lamperti transformation to the time-inhomogeneous case.

3. EXAMPLES AND APPLICATIONS

The goal of this section is to play around with the assumptions of Theorem 2.1. We apply this result to several motivating situations, namely GW processes (Section 3.1), GW processes in varying environment with bounded variance, leading to Feller diffusions in varying environment and with possible jumps (Section 3.2), and finally GW processes with random, i.i.d. environment (Section 3.3).

Some of our limits will be CSBP. To identify CSBP within the framework of Theorem 2.1 we will use the following lemma. For $a \in \mathbb{R}$, $b \geq 0$ and θ a measure on $(0,\infty)$, we call Ξ the branching mechanism with characteristics (a,b,θ) the function satisfying

$$\Xi(\lambda) = a\lambda - b\lambda^2 + \int \left(1 - e^{-\lambda x} - \frac{\lambda x}{1+x^2}\right)\theta(dx),\ \lambda \geq 0.$$

We say that a CSBP has characteristic triplet (a,b,θ) if Ξ is its branching mechanism.

Lemma 3.1. *Fix $a \in \mathbb{R}$, $b \geq 0$ and θ a measure on $(0,\infty)$. Let Ξ be the branching mechanism with characteristics (a,b,θ), i.e.,*

$$\Xi(\lambda) = a\lambda - b\lambda^2 + \int \left(1 - e^{-\lambda x} - \frac{\lambda x}{1+x^2}\right)\theta(dx), \ \lambda \geq 0.$$

For $\lambda > 0$ let u_λ be the unique function satisfying $u_\lambda(t) = \lambda + \int_0^t \Xi(u_\lambda(y))\,dy$ for all $t \geq 0$. Then for each $t, \lambda > 0$, the function $u_{t,\lambda}(s) = u_\lambda(t-s)$ satisfies (3) *for all $0 \leq s \leq t$ with the following choice for α, β and ν: $\alpha(t) = at$, $\beta(t) = bt$ and $\nu(A \times (0,t]) = t\theta(A)$.*

Proof. We prove that $u_{t,\lambda}$ satisfies (11): we have

$$\begin{aligned}\int_{(s,t]} u_{t,\lambda}(y)\alpha(dy) - \int_{(s,t]} (u_{t,\lambda}(y))^2\beta(dy) + \int_{(0,\infty)\times(s,t]} h(x, u_{t,\lambda}(y))\nu(dx\,dy)\\ = a\int_s^t u_\lambda(t-y)dy - b\int_s^t (u_\lambda(t-y))^2 dy + \int_0^\infty \int_s^t h(x, u_\lambda(t-y))\theta(dx)dy\end{aligned}$$

which is equal to $\int_0^{t-s} \Xi(u_\lambda(y))dy = u_\lambda(t-s) - \lambda = u_{t,\lambda}(s) - \lambda$. This proves the result. □

3.1. **Convergence of GW processes.** In this subsection we consider standard GW processes, so that for fixed $n \geq 1$ the offspring distributions $(\mu_{i,n}, i \geq 0)$ are all equal. In particular we have $\alpha_n(t) = \gamma_n(t)\alpha_{0,n}$, $|\alpha_n|(t) = \gamma_n(t)|\alpha_{0,n}|$, $\beta_n(t) = \gamma_n(t)\beta_{0,n}$ and $\nu_n(A \times (0,t]) = \gamma_n(t)\nu_{0,n}(A)$. Note that $|\alpha_n|(t) = |\alpha_n(t)|$ and so we can focus on α_n, β_n and ν_n.

Intuitively, in the homogeneous case it is natural to consider $\gamma_n(t)$ linear in t because the dynamics stays constant over time (this could be rigorously justified). So we write $\gamma_n(t) = \lfloor \vartheta_n t \rfloor$ for some real-valued sequence $(\vartheta_n, n \geq 1)$ going to infinity.

In this case, the assumption (A1) is equivalent to assuming that the functions α, β and $\nu([x,\infty)\times(0,\cdot])$ are linear in t and that

$$\text{(A1')} \qquad \vartheta_n\alpha_{0,n} \to \alpha(1), \ \vartheta_n\beta_{0,n} \to \beta(1) \ \text{ and } \ \vartheta_n\nu_{0,n}([x,\infty)) \to \nu([x,\infty)\times(0,1])$$

as n goes to infinity. In particular, the assumption (A2) is automatically satisfied. We can summarize this as follows.

Corollary 3.2. *In the GW case, if the assumption* (A1') *holds then the sequence (X_n) converges in the sense of finite-dimensional distributions to a CSBP with characteristic triplet (α,β,ν).*

The question is whether this is optimal, i.e., if X_n converges in the sense of finite-dimensional distributions, does (A1') necessarily hold? Grimvall [19, Theorem 3.4] has proved that (X_n) converges in the sense of finite-dimensional distributions if and only if some triangular array converges; combining this with Theorem 1 of § 25 in Gnedenko and Kolmogorov [16], we obtain the following result.

Theorem 3.3 (Theorem 3.4 in [19] and Theorem 1 of § 25 in [16])**.** *If $(X_n(1))$ converges in distribution to a random variable $X(1)$ with $\mathbb{P}(X(1) > 0) > 0$, then there exist $\sigma \geq 0$ and a measure ν_∞ on $(0,\infty)$ such that*

$$\lim_{n\to+\infty} \vartheta_n\nu_{0,n}(\upsilon,\infty) = \nu_\infty(\upsilon,\infty) \tag{6}$$

for every $v>0$ *with* $\nu_\infty\{v\}=0$ *and*

$$\text{(7)}\quad \lim_{\varepsilon\to 0}\limsup_{n\to+\infty}\left\{\vartheta_n\left[\int_{|v|<\varepsilon} v^2\nu_{0,n}(dv)-n^{-1}\left(\int_{|v|<\varepsilon} v\nu_{0,n}(dv)\right)^2\right]\right\}$$
$$=\lim_{\varepsilon\to 0}\liminf_{n\to+\infty}\left\{\vartheta_n\left[\int_{|v|<\varepsilon} v^2\nu_{0,n}(dv)-n^{-1}\left(\int_{|v|<\varepsilon} v\nu_{0,n}(dv)\right)^2\right]\right\}=\sigma^2.$$

Conditions (6) and (7) are sufficient for $(X_n(1))$ to converge in distribution to $X(1)$ with X a CSBP, see Theorem 3.1 in Grimvall [19]. So assuming that $(X_n(1))$ converges in distribution, we see that (6) gives the last part of (A1) concerning the convergence of $\nu_n([x,\infty)\times(0,t])$. Hence it remains to see whether (7) implies $\vartheta_n\alpha_{0,n}\to\alpha(1)$ and $\vartheta_n\beta_{0,n}\to\beta(1)$, i.e., whether (7) implies

$$\lim_{n\to+\infty}\left(\vartheta_n\int\frac{x}{1+x^2}\nu_{0,n}(dx)\right)=\alpha(1)\ \text{ and }\ \lim_{n\to+\infty}\left(\vartheta_n\int\frac{x^2}{2(1+x^2)}\nu_{0,n}(dx)\right)=\beta(1).$$

This holds in the case of Feller diffusion, whereas the general case is left open.

Lemma 3.4. *Let* $\vartheta_n=n$*. If* (X_n) *converges in the sense of finite-dimensional distributions to Feller diffusion, then* (A1') *holds.*

Proof. When (X_n) converges to Feller diffusion and $\vartheta_n=n$, then (6) and (7) become equivalent to

$$n\int v\nu_{0,n}(dv)\underset{n\to+\infty}{\longrightarrow}a,\quad n\int v^2\nu_{0,n}(dv)\underset{n\to+\infty}{\longrightarrow}b\quad\text{and}\quad n\int_{v>\varepsilon}v^2\nu_{0,n}(dv)\underset{n\to+\infty}{\longrightarrow}0$$

for some $a,b\in\mathbb{R}$ and all $\varepsilon>0$, see for instance Chapter 5 in Gnedenko and Kolmogorov [16] or Theorem 3.2 in Grimvall [19]. Hence to show that (A1') holds it is enough to show that

$$\lim_{n\to+\infty}\left(n\left|\int\frac{v^k}{1+v^2}\nu_{0,n}(dv)-\int v^k\nu_{0,n}(dv)\right|\right)=0$$

for $k=1$ or 2. We have

$$n\left|\int\frac{v^k}{1+v^2}\nu_{0,n}(dv)-\int v^k\nu_{0,n}(dv)\right|\le n\int\frac{|v|^{k+2}}{1+v^2}\nu_{0,n}(dv)$$

and since $\nu_{0,n}(-\infty,-1/n)=0$, we get for any $\varepsilon>0$

$$n\int\frac{|v|^{k+2}}{1+v^2}\nu_{0,n}(dv)\le n\frac{n^{-k-2}}{1+n^{-2}}+n\int_{0\le v\le\varepsilon}\frac{v^{k+2}}{1+v^2}\nu_{0,n}(dv)+n\int_{v>\varepsilon}\frac{v^{k+2}}{1+v^2}\nu_{0,n}(dv)$$
$$\le n^{-k-1}+\varepsilon^k n\int v^2\nu_{0,n}(dv)+n\int_{v>\varepsilon}v^2\nu_{0,n}(dv).$$

Hence

$$\limsup_{n\to+\infty}\left(n\left|\int\frac{v^k}{1+v^2}\nu_{0,n}(dv)-\int v^k\nu_{0,n}(dv)\right|\right)\le\varepsilon^k b$$

and letting $\varepsilon\to 0$ gives the result. □

3.2. **Feller diffusion in varying environment.** We prove here that GW processes in varying environment converge to Feller diffusion in varying environment with possible jumps at a fixed time, provided reproduction laws have bounded variance. This result is closely related to Kurtz [25]. Contrarily to [25], we assume here that α has finite variations, but we have weaker moment assumptions and no regularity required for β. This gives a generalization of convergence of GW to Feller diffusion to the case of varying environment which is recalled in the section dedicated to GW case. We note that the limit process

may jump at fixed times. These jumps are multiplicative and may be negative. We refer to [6] for Feller diffusion with multiplicative jumps coming from biological motivations: the jumps correspond to cell division event where only a fraction of parasites is inherited by each daughter cell. The results given here allow to extend the large populations approximations [6] for the parasite population dynamic.

We denote by

$$m_{i,n} = \sum_{k=0}^{n} k\mu_{i,n}\{k\} \text{ and } M_{i,n} = \frac{1}{2n}\sum_{k=0}^{n}(k-1)^2\mu_{i,n}\{k\}.$$

We give here conditions to ensure that (X_n) converges in the sense of finite-dimensional distributions on $[\wp(t), t]$ to a process with Laplace transform described by Theorem 2.1 with $\nu = 0$.

Proposition 3.5. *Assume that there exist a càdlàg function α with locally bounded variations and a non-decreasingcàdlàg function β such that for every $s \geq 0$, $a > 0$, as $n \to \infty$,*

$$\sum_{i=0}^{\lfloor nt \rfloor}\big(m_{i,n}-1, |m_{i,n}-1|, M_{i,n}, n\mu_{i,n}[an,\infty)\big) \to \big(\alpha(t), |\alpha|(t), \beta(t), 0\big). \tag{8}$$

We assume also that for every $t \geq 0$ such that $\alpha\{t\} \neq 0$, $m_{\gamma_n(t),n} - 1 \stackrel{n\to\infty}{\longrightarrow} \alpha\{t\}$.

Then, β is continuous and for all $t \geq 0$, $s \in [\wp(t), t]$ and $\lambda > 0$, $u_n(s,t,\lambda) \to u(s,t,\lambda)$ as $n \to \infty$, where u is the unique solution of the backward differential equation (3) associated to the triplet $(\alpha, \beta, 0)$. More explicitly, we have then

$$u(s,t,\lambda) = \left(\exp(-\bar{\alpha}(s,t])\lambda^{-1} + \int_{(s,t]}\exp(-\bar{\alpha}(s,y])\beta(dy)\right)^{-1},$$

where

$$\bar{\alpha}(t) = \alpha(t) + \sum_{s\leq t}\big[\log(1+\alpha\{s\}) - \alpha\{s\}\big].$$

Under the assumptions of the Proposition above, (X_n) converges in the sense of finite-dimensional distributions to a process denoted by X, which is a Feller diffusion in varying environment whose Laplace exponent is u. Letting λ go to zero and $t \to \infty$ in the explicit expression of u obtained above yields directly the following asymptotic result.

Corollary 3.6. *For every $s \geq 0$, $\mathbb{P}(X_t > 0 \mid X_s = 1) \to 0$ as $t \to \infty$ if and only if*

$$\int_{(s,\infty)}\exp(-\bar{\alpha}(s,y])d\beta(y) = \infty$$

Let us comment these results. The explicit expression of u given above can be guessed in several ways. It can be seen from the discrete expression u_n and explicitly obtained by computing composition of linear fractional probability generating function, see the proof of Lemma 5.2. Also, considering the Laplace exponent of Feller diffusion whose coefficients are constant on successive time intervals and going through the limit gives another intuitive proof of this expression.

Moreover, we note from the proof below that (8) is equivalent to

$$\alpha_n(t) \to \alpha(t), \qquad |\alpha_n(t)| \to |\alpha|(t), \qquad \beta_n(t) \to \beta(t), \quad \nu_n([a,\infty)\times(0,t]) \to 0.$$

We have seen in Section 3.1 that this is the optimal condition to have convergence of GW processes towards Feller diffusion. It is satisfied soon as the second moments of $\mu_{i,n}$ are uniformly bounded.

Proof of Proposition 3.5. First, we use Theorem 2.1 with $\gamma_n(t) = \lfloor nt \rfloor$ to prove that for any $s \in [\wp(t), t]$ we have $u_n(s,t,\lambda) \to u(s,t,\lambda)$ with u is the unique solution of the backward differential equation (3) associated to the triplet $(\alpha, \beta, 0)$. Since $\nu_n([a,\infty) \times (0,t]) = \sum_{i=1}^{\lfloor nt \rfloor} n\mu_{i,n}[an,\infty)$ we get by assumption $\nu_n([a,\infty) \times (0,t]) \to 0$. Moreover for $a \in (0,1)$, as $x/(1+x^2)$ is bounded by some C for $x \geq -1/n$, we have

$$\begin{aligned}|\alpha_{i,n} - (m_{i,n}-1)| \leq &\left| \sum_{k=-1}^{an} \left(\frac{1}{1+(k/n)^2} - 1 \right) k\mu_{i,n}\{k+1\} \right| \\ &+ \sum_{k=an}^{\infty} \left(\frac{k}{1+(k/n)^2} + k \wedge n \right) \mu_{i,n}\{k+1\} \\ \leq &(1 - 1/(1+a^2))|m_{i,n} - 1| + (C+1)\nu_n([a,\infty) \times (0,t]).\end{aligned}$$

Hence the two sequences $(\alpha_{\gamma_n(t),n})$ and $(m_{\gamma_n(t),n})$ must have the same limit, and since $m_{\gamma_n(t),n} \to \alpha\{t\}$ by assumption we obtain $\alpha_{\gamma_n(t),n} \to \alpha\{t\}$. Moreover, summing over i, using $\sup_{n\geq} \sum_{i=0}^{\lfloor nt \rfloor} |m_{i,n} - 1| < \infty$ and letting $a \to 0$, we get

$$\lim_{n\to+\infty} \max\left(\left| \sum_{i=0}^{\lfloor nt \rfloor} (m_{i,n}-1) - \alpha_n(t) \right|, \left| \sum_{i=0}^{\lfloor nt \rfloor} |m_{i,n}-1| - |\alpha_n|(t) \right| \right) = 0$$

and so the assumptions yield $\alpha_n(t) \to \alpha(t)$ and $|\alpha_n|(t) \to |\alpha|(t)$. Similarly we use that $x^2/(1+x^2)$ is bounded by C' for $x \geq -1/n$ so that for $a > 0$,

$$\begin{aligned}|\beta_{i,n} - M_{i,n}| &= \left| \frac{1}{2n} \sum_{k=0}^{\infty} \frac{(k-1)^2}{1+(k-1)^2/n^2} \mu_{i,n}\{k\} - \frac{1}{2n} \sum_{k=0}^{n} (k-1)^2 \mu_{i,n}\{k\} \right| \\ &\leq (1 - 1/(1+a^2)) \frac{1}{2n} \sum_{k=0}^{an} (k-1)^2 \mu_{i,n}\{k\} + [C'+1] \frac{1}{2} \nu_n(t, [a,\infty)).\end{aligned}$$

Then, summing over i, we obtain similarly as before $\beta_n(t) \to \beta(t)$ and $\nu_n([a,\infty) \times (0,t]) \to \nu([a,\infty) \times (0,t])$ for every $t \geq 0$ and $a \geq 0$. It remains to prove that β is continuous to obtain the assumptions (A1) and (A2) of Theorem 2.1 and complete the proof of the convergence of u_n to the unique solution of the backward differential equation (3). To see that, we observe that

$$\begin{aligned}\beta_{i,n} = \frac{1}{2n} \sum_{k=0}^{\infty} \frac{(k-1)^2}{1+(k-1)^2/n^2} \mu_{i,n}\{k\} &\leq \frac{a}{2} \sum_{k=0}^{an} |k-1| \mu_{i,n}\{k\} + C' n\mu_{i,n}[an+1,\infty) \\ &\leq \frac{a}{2}[1 + m_{i,n}] + C' n\mu_{i,n}[an+1,\infty).\end{aligned}$$

Using that $m_{i,n}$ is bounded for $i \leq \gamma_n(t)$ and $n \geq 0$ and that $\mu_{i,n}[an+1,\infty)$ goes to zero uniformly for $i \leq \gamma_n(t)$ by assumption, we deduce that $\beta_{i,n}$ goes uniformly to zero as n goes to infinity by letting a go to zero.

Let us specify now the value of the limit $u(s,t,\lambda)$ of $u_n(s,t,\lambda)$ on $[\wp(t), t]$. We use that α and β have locally finite variations, so the same hold for $s \to \bar{\alpha}(s,t]$ and $s \to I(s) = \int_{(s,t]} \exp(-\bar{\alpha}(s,y])\beta(dy)$. Thus we can simply check by integral computations on functions with finite variations and possible jumps that the function G defined

$$s \to G(s) = F(\bar{\alpha}(s,t], I(s)), \qquad \text{with} \quad F(x,y) = [\exp(-x)/\lambda + y]^{-1}$$

satisfies

$$G(s) = \lambda + \int_{(s,t]} G(y)\alpha(dy) + \int_{(s,t]} G(y)^2 \beta(dy). \tag{9}$$

For that purpose, we distinguish if s is an atom of α or not to get

$$dI(s) = -\exp(-\bar{\alpha}(s,s])\beta(ds) + [\exp(-\bar{\alpha}(ds)) - 1]I(s) = -\beta(ds) + \chi(ds)I(s)$$

where $\chi(ds) = \exp(-\bar{\alpha}(ds)) - 1$, i.e. $\chi(s) = -\alpha(s) + \sum_{u\le s}[\alpha\{s\} + 1/(1+\alpha\{s\}) - 1]$.

Similarly, we compute the jumps of G:

$$\begin{aligned} G\{s\} &= \frac{I\{s\} + [\exp(-\bar{\alpha}\{s\}) - 1]\exp(-\bar{\alpha}(s,t])/\lambda}{[\exp(-\bar{\alpha}(s,t])/\lambda + I(s)][\exp(-\bar{\alpha}\{s\})\exp(-\bar{\alpha}(s,t])/\lambda + I\{s\} + I(s)]} \\ &= \frac{[\exp(-\bar{\alpha}\{s\}) - 1][I(s) + \exp(-\bar{\alpha}(s,t])/\lambda]}{\exp(-\bar{\alpha}\{s\})][\exp(-\bar{\alpha}(s,t])/\lambda + I(s)]^2} \\ &= -\alpha\{s\}G(s) \end{aligned}$$

and get the same jumps as (9). Finally, we note that

$$\frac{\partial F}{\partial x}(x,y) = \frac{\exp(-x)}{\lambda}[\exp(-x)/\lambda + y]^{-2} \qquad \frac{\partial F}{\partial y}(x,y) = -[\exp(-x)/\lambda + y]^{-2}.$$

and derive dG outside the jumps via

$$dG(s) = -\frac{\partial F}{\partial x}(\bar{\alpha}(s,t], I(s))\alpha(ds) + \frac{\partial F}{\partial y}(\bar{\alpha}(s,t), I(s))dI(s),$$

to get (9) and conclude the proof. □

3.3. **Scaling of GW processes in random environment.** The most popular model for GW processes in random environment is when the environments are i.i.d. It has been introduced in [29] and extended to stationary ergodic environments by Athreya and Karlin [3, 4]. We want to study the case where we are mixing J sequences of GW processes with speeds (ϑ_n^j). Our main goal is to gain insight into the correct speed γ_n of the obtained time-varying GW process.

We recall that we work here in the case where drift functions have finite variation; a more general setting has been studied in the case when offspring distributions have finite variance, see, e.g., Kurtz [25]. In this case, the GW processes which are mixed all have the same speed. To the best of our knowledge, the following results when mixing GW processes with different speeds or mixing GW processes with the same speeds but with infinite variance are new. For safe of simplicity of the statements and the proofs, we restrict ourselves to a finite number of environments which occur in an i.i.d. manner, but our approach could be extended to more general cases.

3.3.1. *Notation and assumptions.* In the rest of this section we fix some integer $J \ge 2$. For each $j = 1,\dots,J$, we consider a sequence $(Z^{(n,j)}, n \ge 1)$ of GW processes with corresponding sequence of offspring distributions $(\mu_n^j, n \ge 1)$ and speed $(\vartheta_n^j, n \ge 1)$.

We note $\alpha_{0,n}^j$, $\beta_{0,n}^j$ and $\nu_{0,n}^j$ the numbers and measure defined similarly as $\alpha_{i,n}$, $\beta_{i,n}$ and $\nu_{i,n}$ but with μ_n^j instead of $\mu_{i,n}$, and similarly with the functions and measures α_n^j, β_n^j and ν_n^j (with in addition $\lfloor \vartheta_n^j t \rfloor$ instead of $\gamma_n(t)$). We assume that there exist $\alpha^j \in \mathbb{R}$, $\beta^j \ge 0$ and a measure ν^j such that as $n \to +\infty$, for any $t, x \ge 0$,

$$\alpha_n^j(t) \to t\alpha^j,\ \beta_n^j(t) \to t\beta^j \text{ and } \nu_n^j([x,\infty) \times (0,t]) \to t\nu^j([x,\infty)).$$

In particular, the assumptions (A1) and (A2) are satisfied for the jth sequence of GW processes $(Z^{(n,j)}, n \ge 1)$ with speed (ϑ_n^j), which converges to a CSBP with characteristics $(\alpha^j, \beta^j, \nu^j)$.

We now consider the case where we are mixing these J GW processes in the simplest way. To do so, we assume that for each $n \geq 1$, the offspring distributions $(\mu_{i,n}, i \geq 0)$ defining $Z^{(n)}$ are i.i.d. with $\mu_{i,n} = \mu_n^j$ with probability $p^j > 0$. Let $N_{k,n}^j$ be the number of times the jth environment has been chosen among the k first generations in the nth branching process, then we have conditionally on the environment

$$\alpha_n(t) = \sum_{i=0}^{\gamma_n(t)-1} \alpha_{i,n} = \sum_{j=1}^J N_{\gamma_n(t),n}^j \alpha_{0,n}^j = \sum_{j=1}^J \overline{N}_{\gamma_n(t),n}^j \frac{p^j \gamma_n(t)}{\lfloor \vartheta_n^j t \rfloor} \alpha_n^j(t)$$

where $\overline{N}_{k,n}^j = N_{k,n}^j / (kp^j)$. Note that the law of large numbers suggests that $\overline{N}_{\gamma_n(t),n}^j \approx 1$, an approximation to which we will come back shortly. We get similarly

$$\beta_n(t) = \sum_{j=1}^J \overline{N}_{\gamma_n(t),n}^j \frac{p^j \gamma_n(t)}{\lfloor \vartheta_n^j t \rfloor} \beta_n^j(t)$$

and

$$\nu_n([x,\infty) \times (0,t]) = \sum_{j=1}^J \overline{N}_{\gamma_n(t),n}^j \frac{p^j \gamma_n(t)}{\lfloor \vartheta_n^j t \rfloor} \nu_n^j([x,\infty) \times (0,t]).$$

To satisfy the assumptions (A1) and (A2), we need the almost sure convergence of α_n, β_n and ν_n and so we need to build the processes $Z^{(n)}$ on the same probability space. A way to do so is to consider $\pi^0 = 0$, $\pi^j = p^1 + \cdots + p^j$ for $1 \leq j \leq J$, $(U_i, i \geq 1)$ a sequence of i.i.d. random variables uniformly distributed on $[0,1]$ and to let

$$N_{k,n}^j = \sum_{i=1}^k \mathbb{1}_{\{\pi^{j-1} \leq U_i \leq \pi^j\}}.$$

Then $N_{k,n}^j$ does not depend on n and the strong law of large numbers immediately implies that for every $t > 0$, $\overline{N}_{\gamma_n(t),n}^j = \overline{N}_{\gamma_n(t)}^j \to 1$ almost surely, as n goes to infinity.

3.3.2. *Around* γ_n. When we observe X_n in $[0,t]$ we see in average $p^j \gamma_n(t)$ times the jth environment. For the jth GW process to evolve significantly, we need to observe it over on the order of at least ϑ_n^j generations. Hence if $\gamma_n(t) \ll \vartheta_n^j$ for each j, then we expect X_n to have not evolved at all. On the other hand, if $\gamma_n(t) \gg \vartheta_n^j$ for some j, then we expect the jth GW process to have already reached its terminal value. This latter case can be subtle, but this shows that when mixing environments, the speed that dominates is the speed of the "fastest" GW process, i.e., the GW process with speed $\vartheta_n^* = \min_j \vartheta_n^j$ (we call it the fastest because this is the GW that needs to be sped up by the smallest speed). The following simple result captures this intuition.

Lemma 3.7. *If* $\sup_{n \geq 1} \left(\gamma_n(t) / \vartheta_n^* \right) < +\infty$, *then*

$$\lim_{n \to +\infty} \left| \alpha_n(t) - \sum_{j=1}^J \frac{p^j \gamma_n(t)}{\vartheta_n^j} \alpha^j \right| = 0$$

for every $t \geq 0$, *and similarly with* $|\alpha|$, β *and* ν.

Proof. The result is obvious for $t = 0$, so consider $t > 0$. We have

$$\begin{aligned}\left|\alpha_n(t) - \sum_{j=1}^J \frac{p^j \gamma_n(t)}{\vartheta_n^j}\alpha^j\right| &= \left|\sum_{j=1}^J \overline{N}^j_{\gamma_n(t),n} \frac{p^j\gamma_n(t)}{\lfloor \vartheta_n^j t\rfloor}\alpha_n^j(t) - \sum_{j=1}^J \frac{p^j\gamma_n(t)}{\vartheta_n^j}\alpha^j\right| \\ &\le \sum_{j=1}^J \frac{p^j\gamma_n(t)}{\vartheta_n^j}\left|\alpha^j - \overline{N}^j_{\gamma_n(t)}\frac{\vartheta_n^j t}{\lfloor\vartheta_n^j t\rfloor}\frac{\alpha_n^j(t)}{t}\right| \\ &\le \sup_{k\ge 1}\left(\frac{\gamma_k(t)}{\vartheta_n^*}\right)\times \max_{1\le j\le J}\left|\alpha^j - \overline{N}^j_{\gamma_n(t)}\frac{\vartheta_n^j t}{\lfloor\vartheta_n^j t\rfloor}\frac{\alpha_n^j(t)}{t}\right|\end{aligned}$$

which goes to 0 as n goes to infinity since the first term of the last upper bound is finite by assumption, $\alpha_n^j(t)/t \to \alpha^j$ also by assumption and $\overline{N}^j_{\gamma_n(t)} \to 1$ by construction. This proves the result. □

In particular, the assumption (A1) holds with all limits α, β and ν being degenerate (null) when $\gamma_n(t) \ll \vartheta_n^*$ for each $t \ge 0$. In the following subsection we will investigate more interesting cases, when the limit is not degenerate, but before that let us discuss a natural choice for γ_n.

When the environment in the ith generation is equal to μ_n^j, it is natural to want to *locally* speed up time by ϑ_n^j between the times t_i^n and t_{i+1}^n; this amounts to choose $\gamma_n(t)$ in such a way that $t_{i+1}^n - t_i^n = 1/\vartheta_n^j$ if $\mu_{i,n} = \mu_n^j$. This leads to define $\vartheta_{i,n} = \vartheta_n^j$ if $\mu_{i,n} = \mu_n^j$ and to consider the function

$$\gamma_n^{\mathrm{loc}}(t) = \inf\left\{k \ge 1 : \sum_{i=0}^k \frac{1}{\vartheta_{i,n}} \ge t\right\}.$$

By definition,

$$\sum_{i=0}^{\gamma_n^{\mathrm{loc}}(t)-1}\frac{1}{\vartheta_{i,n}} < t \le \sum_{i=0}^{\gamma_n^{\mathrm{loc}}(t)}\frac{1}{\vartheta_{i,n}} \iff t \le \sum_{i=0}^{\gamma_n^{\mathrm{loc}}(t)}\frac{1}{\vartheta_{i,n}} < t + \frac{1}{\gamma_{\gamma_n^{\mathrm{loc}}(t),n}}$$

and since

$$\sum_{i=0}^{\gamma_n^{\mathrm{loc}}(t)}\frac{1}{\vartheta_{i,n}} = \sum_{j=1}^J \frac{N^j_{\gamma_n^{\mathrm{loc}}(t)}}{\vartheta_n^j} = \gamma_n^{\mathrm{loc}}(t)\sum_{j=1}^J \overline{N}^j_{\gamma_n^{\mathrm{loc}}(t)}\frac{p^j}{\gamma_n^j}$$

we obtain

$$\lim_{n\to+\infty}\left(\gamma_n^{\mathrm{loc}}(t)\sum_{j=1}^J\frac{p^j}{\vartheta_n^j}\right) = t.$$

In other words, we have $\gamma_n^{\mathrm{loc}}(t) \approx \Gamma_n t$ for large n, where

$$\Gamma_n = \left(\frac{p^1}{\vartheta_n^1} + \cdots + \frac{p^J}{\vartheta_n^J}\right)^{-1}.$$

3.3.3. *Two extreme cases.* We use this result to show convergence of X_n in two extreme cases, when all the speeds are equal or when in contrast one speed dominates the others.

Proposition 3.8 (Mixing of GW with same speeds)**.** *Assume that $\vartheta_n^j = \vartheta_n^1$ for all $j = 1,\dots,J$ and let either $\gamma_n(t) = \lfloor\vartheta_n^1 t\rfloor$ or $\gamma_n = \gamma_n^{loc}$. Then (X_n) converges in the sense of finite-dimensional distributions to the CSBP with characteristics $(\sum_j p^j\alpha^j, \sum_j p^j\beta^j, \sum_j p^j\nu^j)$.*

Proof. Under the assumptions of the lemma, we have $\vartheta_n^* = \vartheta_n^1 = \Gamma_n$ so $\gamma_n(t) \sim \vartheta_n^1 t$ as n goes to infinity. In particular, $\sup_{n\geq 1}\left(\gamma_n(t)/\vartheta_n^*\right) < +\infty$ and Lemma 3.7 gives

$$\lim_{n\to+\infty} \alpha_n(t) = \lim_{n\to+\infty} \sum_{j=1}^{J} \frac{p^j \gamma_n(t)}{\vartheta_n^1} \alpha^j = t \sum_{j=1}^{J} p^j \alpha^j.$$

Similar computations hold for $|\alpha|$, β and ν and so the assumptions (A1) and (A2) hold with $\alpha(t) = (p^1\alpha^1 + \cdots + p^j\alpha^j)\,t$ and corresponding linear functions for β and ν. Applying Lemma 3.1 gives the result. $\square$

Proposition 3.9 (Mixing of GW with different speeds)**.** *Assume that $\vartheta_n^1 \ll \vartheta_n^j$ for every $2 \leq k \leq J$ and choose $\gamma_n(t) = \lfloor \vartheta_n^1 t \rfloor$ (resp. $\gamma_n = \gamma_n^{loc}$). Then (X_n) converges in the sense of finite-dimensional distributions to the CSBP with characteristics $(p^1\alpha^1, p^1\beta^1, p^1\nu^1)$ (resp. $(\alpha^1, \beta^1, \nu^1)$).*

Proof. Under the assumptions of the lemma, we have $\vartheta_n^* = \vartheta_n^1$ and $\gamma_n(t) \sim \vartheta_n^1 t$ (resp. $\gamma_n(t) \sim \vartheta_n^1 t/p^1$) as n goes to infinity, when $\gamma_n(t) = \lfloor \vartheta_n^1 t \rfloor$ (resp. $\gamma_n = \gamma_n^{\mathrm{loc}}$). Thus in both cases we have $\sup_n(\gamma_n(t)/\vartheta_n^*) < \infty$ and so Lemma 3.7 gives

$$\lim_{n\to+\infty} \alpha_n(t) = \lim_{n\to+\infty} \sum_{j=1}^{J} \frac{p^j \gamma_n(t)}{\vartheta_n^j} \alpha^j = t p^1 \alpha^1$$

when $\gamma_n(t) = \lfloor \vartheta_n^1 t \rfloor$ and $\alpha_n(t) \to t\alpha^1$ when $\gamma_n = \gamma_n^{\mathrm{loc}}$. Similar computations hold for $|\alpha_n|$, β_n and ν_n and we conclude as in the proof of the previous lemma. $\square$

Remark 3.10. The above results could be extended to a more general case where also the probabilities $p^j = p_n^j$ are allowed to depend on n. If they don't vanish, i.e., $p_n^j \to p^j \in (0,1)$ for each $j = 1,\ldots,J$ then the above results remain true. If $p_n^j \to 0$ for some j, then what matters is not the speed ϑ_n^j but the ratio ϑ_n^j/p_n^j. Indeed, in $[0,t]$ we see in average $p_n^j \gamma_n(t)$ times the jth environment, which similarly as before needs to be compared to ϑ_n^j.

3.4. **Remarks on CSBP with catastrophes.** Theorem 2.1 makes it possible to study GW processes where only few offspring distributions are not near-critical. The simplest example is given by taking all the $\mu_{i,n}$'s equal to a critical offspring distribution μ_n, in such a way that the corresponding GW processes would converge to a CSBP. Then one can change $\mu_{\gamma_n(t),n}$ and take its mean equal to $1+\alpha\{t\}$. Then (X_n) would converge to a process X which is a CSBP on $[0,t)$ and on $[t,\infty)$ and such that $X(t) = (1+\alpha\{t\})X(t-)$. Another way to create a discontinuity at a fixed time is to take $\mu_{\gamma_n(t),n} = (1-1/n)\delta_0 + (1/n)\delta_n$ with δ_a the Dirac mass at a. Again, (X_n) would converge to a process X which is a CSBP on $[0,t)$ and on $[t,\infty)$ and such that $X(t) = S(X(t-))$ with $(S(t), t\geq 0)$ a Poisson process. Theorem 2.1 allows accumulation of such fixed jumps; note that in both cases these jumps may be negative, whereas time-homogeneous CSBP only have positive jumps.

Building up on these two simple examples, we expect in general that if X is a Markov process, possibly time-inhomogeneous, satisfying the branching property and with a fixed discontinuity at time $t \geq 0$, then there should exist a subordinator S_t such that $X(t) = S_t(X(t-))$. Indeed, preliminary results suggest that the Markov property should imply the existence of such a process S_t, while the branching property of X would force S_t to be a subordinator.

4. Proof of Theorem 2.1 and Corollary 2.3

The following functions g and h will be used repeatedly in the sequel:

$$g(x,\lambda) = 1 - e^{-\lambda x} - \frac{\lambda x}{1+x^2} \ \text{ and } \ h(x,\lambda) = g(x,\lambda) + \frac{(\lambda x)^2}{2(1+x^2)}, \ x \in \mathbb{R}, \lambda \geq 0. \tag{10}$$

Theorem 2.1 and Corollary 2.3 are proved in Sections 4.4 and 4.5. Before that, we give an overview of the proof in Section 4.1, where we also introduce additional notation. In Section 4.2 we establish preliminary results, used in Section 4.3 to get uniform controls on u_n: Lemmas 4.3 and 4.5 prove that u_n is bounded away from 0 and infinity and Lemma 4.4 gives a control on the variations of $u_n(\cdot, t, \lambda)$. These controls are used in Section 4.4 to prove Theorem 2.1 via Gronwall type arguments, and in Section 4.5 to prove Corollary 2.3.

4.1. **Overview of the proof and additional notation.**

Recalling the measures α, β, $\widetilde{\beta}$ and ν defined in the statement of Theorem 2.1 and the function h defined in (10), one sees that (3) can be rewritten in the following form:

$$\begin{aligned} u(s,t,\lambda) = \lambda + \int_{(s,t]} u(y,t,\lambda)\alpha(dy) - \int_{(s,t]} (u(y,t,\lambda))^2 \beta(dy) \\ + \int_{(0,\infty)\times(s,t]} h(x, u(y,t,\lambda))\nu(dx\,dy). \end{aligned} \tag{11}$$

This expression will turn out to be technically convenient because of the behavior of $h(x,\lambda)$ as $x \to 0$. Roughly speaking, $h(x,\lambda)$ goes fast enough to zero as $x \to 0$ to make get ride of the indetermination as $n \to \infty$ and let the third term converge, see Lemmas 4.8 and A.2. We now derive a similar dynamics for u_n. Define from now on $\psi_{i,n}$ the function

$$\psi_{i,n}(\lambda) = -n \log\left(1 - \frac{1}{n}\int \left(1 - e^{-\lambda x}\right)\nu_{i,n}(dx)\right), \ \lambda \geq 0. \tag{12}$$

The functions $\psi_{i,n}$ define the dynamics of u_n via the following recursion.

Lemma 4.1. *For any $n \geq 1$, $\lambda \geq 0$ and $0 \leq s \leq t$, it holds that*

$$u_n(s,t,\lambda) = \lambda + \sum_{i=\gamma_n(s)+1}^{\gamma_n(t)} \psi_{i-1,n}(u_n(t_i^n, t, \lambda)). \tag{13}$$

Proof. By definitions of $\psi_{i,n}$ and v_n, we have $v_n(i,i+1,\lambda) = n^{-1}\psi_{i,n}(n\lambda) + \lambda$. Using the Markov and branching properties, we get the following composition rule:

$$v_n(i,k,\lambda) = v_n(i,j,v_n(j,k,\lambda)), \ i \leq j \leq k$$

and in particular,

$$v_n(i,j,\lambda) - v_n(i+1,j,\lambda) = v_n(i,i+1,v_n(i+1,j,\lambda)) - v_n(i+1,j,\lambda) = \frac{1}{n}\psi_{i,n}(n v_n(i+1,j,\lambda)).$$

Since $n v_n(\gamma_n(t), \gamma_n(t), \lambda/n) = \lambda$, this gives

$$\begin{aligned} u_n(s,t,\lambda) &= n v_n(\gamma_n(s), \gamma_n(t), \lambda/n) \\ &= \lambda + n \sum_{i=\gamma_n(s)}^{\gamma_n(t)-1} \left[v_n(i, \gamma_n(t), \lambda/n) - v_n(i+1, \gamma_n(t), \lambda/n) \right] \\ &= \lambda + \sum_{i=\gamma_n(s)}^{\gamma_n(t)-1} \psi_{i,n}(n v_n(i+1, \gamma_n(t), \lambda/n)), \end{aligned}$$

which proves the result, plugging in the relation $u_n(s,t,\lambda) = n v_n(\gamma_n(s), \gamma_n(t), \lambda/n)$ and recalling $\gamma_n(t_i^n) = i$. □

Let us now explain how to go from (13) to (11). Because of the factor $1/n$, under reasonable assumptions the term $(1/n)\int \left(1-e^{-\lambda x}\right)\nu_{i,n}(dx)$ appearing in the definition (12) of $\psi_{i,n}$ should be small for large n. Then the approximation $-\log(1-x) \approx x$ for small x suggests that

$$\psi_{i,n}(\lambda) \approx \int \left(1-e^{-\lambda x}\right)\nu_{i,n}(dx) = \lambda\alpha_{i,n} - \lambda^2\beta_{i,n} + \int_{(0,\infty)} h(x,\lambda)\nu_{i,n}(dx)$$

where the last equality follows from the definitions of $\alpha_{i,n}$, $\beta_{i,n}$ and h. In combination with (13), this last approximation suggests that

$$\begin{aligned} u_n(s,t,\lambda) \approx \lambda + \sum_{i\geq 1} \mathbb{1}_{\{s<t_i^n\leq t\}} u_n(t_i^n,t,\lambda)\alpha_{i-1,n} - \sum_{i\geq 1} \mathbb{1}_{\{s<t_i^n\leq t\}} (u_n(t_i^n,t,\lambda))^2\beta_{i-1,n} \\ + \sum_{i\geq 1} \mathbb{1}_{\{s<t_i^n\leq t\}} \int_{(0,\infty)} h(x, u_n(t_i^n,t,\lambda))\nu_{i-1,n}(dx) \end{aligned}$$

which can be rewritten as follows, remembering the definitions of the measures α_n, β_n and ν_n:

$$\begin{aligned} u_n(s,t,\lambda) \approx \lambda + \int_{(s,t]} u_n(y,t,\lambda)\alpha_n(dy) - \int_{(s,t]} (u_n(y,t,\lambda))^2\beta_n(dy) \\ + \int_{(0,\infty)\times(s,t]} h(x,u_n(y,t,\lambda))\nu_n(dx\,dy). \end{aligned}$$

This last approximation, combined with the convergence of the triplet $(\alpha_n, \beta_n, \nu_n)$ to (α, β, ν) in the sense of the assumption (A1), suggests that any limit $u(s,t,\lambda)$ of the sequence $(u_n(s,t,\lambda))$ should satisfy the dynamics (11). Let us conclude this section by commenting on the branching mechanism in the time-inhomogeneous case and by explaining how to get the finite-dimensional convergence of Corollary 2.3; this will be the opportunity to introduce additional notation which will be used in the following sections.

In the time-homogeneous case, u is characterized by the branching mechanism ψ via the equation

$$u(t,\lambda) = \lambda + \int_0^t \psi(u(x,\lambda))\,dx,$$

see the introduction for more details. In the time-inhomogeneous case, the new dynamics (3) suggests that the branching mechanism becomes in some sense a measure-valued mapping. Indeed, defining for every measurable, positive function $f : [0,\infty) \to (0,\infty)$ the measure $\Psi(f)$ via

$$\Psi(f)(A) = \int_A f\,d\alpha - \int_A f^2\,d\beta + \int_{(0,\infty)\times A} h(x,f(y))\nu(dx\,dy),\ A \in \mathscr{B}, \tag{14}$$

we see that (3) can be rewritten as

$$u(s,t,\lambda) = \lambda + \Psi(u(\cdot,t,\lambda))((s,t]) = \lambda + \int_{(s,t]} \Psi(u(\cdot,t,\lambda))(dx). \tag{15}$$

In analogy with Ψ, we also define for each $n \geq 1$ the measure $\Psi_n(f)$ as follows:

$$\Psi_n(f)(A) = \sum_{i\geq 1} \mathbb{1}_{\{t_i^n\in A\}} \psi_{i-1,n}(f(t_i^n)),\ A \in \mathscr{B}, \tag{16}$$

so that (13) becomes equivalent to

$$u_n(s,t,\lambda) = \lambda + \Psi_n(u_n(\cdot,t,\lambda))((s,t]). \tag{17}$$

Note that Ψ and Ψ_n have been defined for functions $f : [0,\infty) \to (0,\infty)$. In the sequel, with an abuse of notation we will also consider $\Psi(f)$ and $\Psi_n(f)$ for functions f only defined on a subset of $[0,\infty)$, typically $[\wp(t), t]$. Then we will only consider $\Psi(f)(A)$ or $\Psi_n(f)(A)$ for Borel sets A which are subset of the domain of definition of f.

Let us finally comment on the proof of Corollary 2.3. Thanks to the Markov and branching properties, we have for any $\lambda_1, \lambda_2 > 0$ and $s \le t_1 \le t_2 \le t$

$$\begin{aligned}\mathbb{E}\left(e^{-\lambda_1 X_n(t_1) - \lambda_2 X_n(t_2)} \mid X_n(s) = 1\right) &= \mathbb{E}\left[e^{-\lambda_1 X_n(t_1)}\mathbb{E}\left(e^{-\lambda_2 X_n(t_2)} \mid X_n(t_1)\right) \mid X_n(s) = 1\right]\\ &= \mathbb{E}\left[e^{-\lambda_1 X_n(t_1)} e^{-X_n(t_1) u_n(t_1,t_2,\lambda_2)} \mid X_n(s) = 1\right]\end{aligned}$$

and so

$$\mathbb{E}\left(\exp\left(-\lambda_1 X_n(t_1) - \lambda_2 X_n(t_2)\right) \mid X_n(s) = 1\right) = \exp\left(-u_n(s,t_1,\lambda_1 + u_n(t_1,t_2,\lambda_2))\right). \tag{18}$$

Hence to prove convergence of the finite-dimensional distributions, we need a stronger result than the convergence $u_n(s,t,\lambda) \to u(s,t,\lambda)$ for fixed λ, which is the content of Theorem 2.1. Namely, we need to show that $u_n(s,t,\ell_n) \to u(s,t,\lambda)$ if (ℓ_n) is a sequence converging to λ. This explains why in Section 4.4 we will derive such convergence results, which are stronger than what needed for Theorem 2.1. However, because $u_n(s,t,\lambda)$ is increasing in λ these stronger results will come almost for free from the results for fixed λ derived in Sections 4.2 and 4.3.

4.2. **Preliminary results.** Defining for $x \in \mathbb{R}$

$$\Phi_1(x) = \frac{e^{-x} - 1 + x}{x^2} \ \text{ and } \ \Phi_2(x) = \frac{-e^{-x} + 1 - x + x^2/2}{x^2},$$

with $\Phi_1(0) = 1/2$ and $\Phi_2(0) = 0$, we can rewrite g and h, defined in (10), as

$$g(x,\lambda) = \frac{x^2}{1+x^2}\left(1 - e^{-\lambda x} - \lambda^2 \Phi_1(\lambda x)\right) \text{ and } h(x,\lambda) = \frac{x^2}{1+x^2}\left(1 - e^{-\lambda x} + \lambda^2 \Phi_2(\lambda x)\right). \tag{19}$$

Since $\lim_{x\to+\infty} \Phi_1(x) = 0$, Φ_1 is bounded on $[-C,+\infty)$ for any $C > 0$ and in particular, the constant

$$c_1'(C) = \sup\left\{\frac{2|g(x,\lambda)|(1+x^2)}{x^2} : x \ge -1, 0 \le \lambda \le C\right\}$$

is finite. We have by definition

$$\int \left(1 - e^{-\lambda x}\right) \nu_{i,n}(dx) = \lambda \alpha_{i,n} + \int g(x,\lambda) \nu_{i,n}(dx)$$

and since $|g(x,\lambda)| \le c_1'(C) x^2/(2(1+x^2))$ for all $x \ge -1$ and $0 \le \lambda \le C$ by definition of c_1', we get

$$\sup_{0\le\lambda\le C}\left|\int \left(1 - e^{-\lambda x}\right) \nu_{i,n}(dx)\right| \le C|\alpha_{i,n}| + c_1'(C)\beta_{i,n} \le c_1(C)\mu_n(t_i^n, t_{i+1}^n] \tag{20}$$

where from now on $c_1(C) = C + c_1'(C)$ and $\mu_n = |\alpha_n| + \beta_n$, i.e.,

$$\mu_n(A) = |\alpha_n|(A) + \beta_n(A), \ A \in \mathscr{B}.$$

In Section 4.1 it is explained that one of the key step to prove Theorem 2.1 lies in the approximation

$$\psi_{i,n}(\lambda) \approx \int \left(1 - e^{-\lambda x}\right) \nu_{i,n}(dx).$$

To justify this approximation, we introduce in the sequel $\epsilon_{i,n}$ the function such that for any $\lambda \geq 0$,

$$\psi_{i,n}(\lambda) = \left(1 + \epsilon_{i,n}(\lambda)\right) \int \left(1 - e^{-\lambda x}\right) \nu_{i,n}(dx).$$

When $\int (1 - e^{-\lambda x}) \nu_{i,n}(dx) = 0$, we set $\epsilon_{i,n}(\lambda) = 0$. In the sequel, for $n \geq 1$ and $t, C \geq 0$ we define the constant $\overline{c}^{\epsilon}_{n,t}(C)$ as:

$$\overline{c}^{\epsilon}_{n,t}(C) = \sup\left\{|\epsilon_{i,n}(\lambda)| : 0 \leq i < \gamma_n(t), 0 \leq \lambda \leq C\right\}. \tag{21}$$

Lemma 4.2. *Fix $t \geq 0$ and assume that the two sequences $(|\alpha_n|(t))$ and $(\beta_n(t))$ are bounded. Then for any $C \geq 0$, we have $\overline{c}^{\epsilon}_{n,t}(C) \to 0$ as n goes to infinity.*

Proof. Fix t and $C \geq 0$ and note

$$I_{t,C} = \sup\left\{\left|\int (1 - e^{-\lambda x}) \nu_{i,n}(dx)\right| : 1 \leq n, 0 \leq i < \gamma_n(t), 0 \leq \lambda \leq C\right\}.$$

Then (20) entails

$$\begin{aligned} I_{t,C} &\leq c_1(C) \sup\left\{\mu_n(t^n_i, t^n_{i+1}] : n \geq 1, 0 \leq i < \gamma_n(t)\right\} \\ &\leq c_1(C) \sup_{n \geq 1} \left(\sum_{i=0}^{\gamma_n(t)-1} \mu_n(t^n_i, t^n_{i+1}] \right) = c_1(C) \sup_{n \geq 1} \mu_n(t). \end{aligned}$$

Since $\mu_n(t) = |\alpha_n|(t) + \beta_n(t)$ the sequence $(\mu_n(t))$ is bounded by assumption, showing that $I_{t,C}$ is finite. It follows from the definition of $\psi_{i,n}$ and $\epsilon_{i,n}$ that for any $i \geq 0$

$$\epsilon_{i,n}(\lambda) = \frac{-\log\left(1 - \frac{1}{n}\int (1 - e^{-\lambda x}) \nu_{i,n}(dx)\right) - \frac{1}{n}\int (1 - e^{-\lambda x}) \nu_{i,n}(dx)}{\frac{1}{n}\int (1 - e^{-\lambda x}) \nu_{i,n}(dx)}$$

and so

$$\overline{c}^{\epsilon}_{n,t}(C) \leq \sup_{|x| \leq I_{t,C}/n} \left| \frac{-\log(1-x) - x}{x} \right|$$

and the result follows by letting $n \to +\infty$. □

4.3. Uniform controls on u_n.

In the sequel for $t, \lambda \geq 0$ we define the constant

$$\overline{c}^{u}_{t,\lambda} = \sup\left\{u_n(s, t, \lambda) : n \geq 1, 0 \leq s \leq t\right\}. \tag{22}$$

The following result ensures that u does not explode in finite time.

Lemma 4.3. *Fix $t \geq 0$ and assume that the two sequences $(|\alpha_n|(t))$ and $(\beta_n(t))$ are bounded. Then for any $\lambda \geq 0$ the constant $\overline{c}^{u}_{t,\lambda}$ is finite and moreover*

$$\sup\left\{\overline{c}^{u}_{s,\lambda} : 0 \leq s \leq t, \lambda \leq 1\right\} < \infty.$$

Proof. In the rest of the proof fix t and $\lambda \geq 0$, note $B_t = 2\sup_{n \geq 1} \mu_n(t)$, which is finite by assumption, and $C_{t,\lambda} = (\lambda + 2)(1 + B_t)e^{B_t}$. Following Lemma 4.2 choose $n_{t,\lambda} \geq 1$ such that $\overline{c}^{\epsilon}_{n,t}(C_{t,\lambda}) \leq 1$ for all $n \geq n_{t,\lambda}$. Since $Z^{(n)}_i$ is finite for each $i \geq 0$ and $n \geq 1$, it follows that $\sup_{0 \leq s \leq t} u_n(s, t, \lambda)$ is finite for each $n \geq 1$ and so to prove the result, it is enough to prove that

$$\sup\left\{u_n(s, t, \lambda) : n \geq n_{t,\lambda}, 0 \leq s \leq t\right\} = \sup\left\{u_n(t^n_i, t^n_{\gamma_n(t)}, \lambda) : n \geq n_{t,\lambda}, 0 \leq i \leq \gamma_n(t)\right\}$$

is finite. In the rest of the proof fix $n \geq n_{t,\lambda}$ and note $a_i = u_n(t_i^n, t_{\gamma_n(t)}^n, \lambda)$. We prove by backwards induction that $a_i \leq C_{t,\lambda}$ for all $0 \leq i \leq \gamma_n(t)$, and since the bound does not depend on n or i this will show the result. We have $a_{\gamma_n(t)} = \lambda \leq C_{t,\lambda}$ so the initialization is satisfied. Now consider some $1 \leq i < \gamma_n(t)$ and assume that $a_k \leq C_{t,\lambda}$ for all $i \leq k \leq \gamma_n(t)$: we prove that $a_{i-1} \leq C_{t,\lambda}$.

Fix some $i < k \leq \gamma_n(t)$. By definition, we have

$$\psi_{k-1,n}(a_k) = (1+\epsilon_{k-1,n}(a_k))\left(a_k\alpha_{k-1,n} + \int g(x,a_k)\nu_{k-1,n}(dx)\right).$$

By induction hypothesis, it holds that $a_k \leq C_{t,\lambda}$. Combined with $\overline{c}^{\epsilon}_{n,t}(C_{t,\lambda}) \leq 1$ (since $n \geq n_{t,\lambda}$), this gives $0 \leq 1+\epsilon_{k-1,n}(a_k) \leq 2$. Together with the inequality $g(x,y) \leq x^2/(1+x^2)$, which holds for all $x, y \in \mathbb{R}$ because $\Phi_1 \geq 0$ by convexity, see (19), we finally get

$$\psi_{k-1,n}(a_k) \leq (1+\epsilon_{k-1,n}(a_k))\left(a_k|\alpha_{k-1,n}| + 2\beta_{k-1,n}\right) \leq 2(a_k+2)\mu_n(t_{k-1}^n, t_k^n].$$

Hence for any $i-1 \leq j \leq \gamma_n(t)$, this gives together with Lemma 4.1 for the first equality

$$a_j = \lambda + \sum_{k=j+1}^{\gamma_n(t)} \psi_{k-1,n}(a_k) \leq \lambda + \sum_{k=j+1}^{\gamma_n(t)} 2(a_k+2)\mu_n(t_{k-1}^n, t_k^n].$$

This can be rewritten $a'_j \leq A + S_{j+1}$ if $a'_k = a_k + 2$, $A = \lambda + 2$, $d_k = 2\mu_n(t_{k-1}^n, t_k^n]$ and $S_k = d_k a'_k + \cdots + d_{\gamma_n(t)} a'_{\gamma_n(t)}$. This gives for $j = i-1$

$$a'_{i-1} \leq A + S_i = A + d_i a'_i + S_{i+1} \leq A + d_i(A+S_{i+1}) + S_{i+1} = (1+d_i)(A+S_{i+1}).$$

Then by induction one gets

$$a'_{i-1} \leq (1+d_i)\cdots(1+d_{\gamma_n(t)-1})(A+S_{\gamma_n(t)}) \leq \exp\left(d_1+\cdots+d_{\gamma_n(t)}\right)(A+d_{\gamma_n(t)}a'_{\gamma_n(t)}).$$

Since $a'_{\gamma_n(t)} = A = \lambda+2$ and $d_{\gamma_n(t)} \leq d_1 + \cdots + d_{\gamma_n(t)} = 2\mu_n(t) \leq B_t$, this shows that $a_{i-1} \leq C_{t,\lambda}$ which achieves the proof of the induction and shows that $\overline{c}^u_{t,\lambda} \leq C_{t,\lambda}$. This gives the finiteness of $\overline{c}^u_{t,\lambda}$. And since $C_{t,\lambda}$ is clearly increasing in both t and λ, for any $s \leq t$ and $\lambda \geq 1$ we obtain $\overline{c}^u_{s,t} \leq C_{t,1}$ which gives the second part of the lemma. □

In the sequel for $t, \lambda \geq 0$ we define $\Delta^u_{t,\lambda}$ by

$$\Delta^u_{t,\lambda} = \left(1 + \sup_{n\geq 1}\left\{\overline{c}^{\epsilon}_{n,t}\left(\overline{c}^u_{t,\lambda}\right)\right\}\right) c_1\left(\overline{c}^u_{t,\lambda}\right). \tag{23}$$

Note that $\Delta^u_{t,\lambda}$ is finite in view of Lemmas 4.2 and 4.3 when the two sequences $(|\alpha_n|(t))$ and $(\beta_n(t))$ are bounded.

Lemma 4.4. *The inequality*

$$\left|u_n(s,t,\lambda) - u_n(s',t,\lambda)\right| \leq \Delta^u_{t,\lambda}\mu_n(s,s']$$

holds for all $0 \leq s \leq s' \leq t$ *and* $\lambda > 0$.

Proof. Let $0 \leq s \leq s' \leq t$ and $\lambda > 0$: Lemma 4.1 and the definition of $\epsilon_{i,n}$ give

$$\begin{aligned}&\left|u_n(s,t,\lambda) - u_n(s',t,\lambda)\right| \\ &\qquad \leq \sum_{i=\gamma_n(s)+1}^{\gamma_n(s')} \left(1 + \left|\epsilon_{i-1,n}(u_n(t_i^n,t,\lambda))\right|\right)\left|\int (1-e^{-xu_n(t_i^n,t,\lambda)})\nu_{i-1,n}(dx)\right|.\end{aligned}$$

Recall the definitions (21) and (22) of $\overline{c}^{\epsilon}_{n,t}(C)$ and $\overline{c}^{u}_{n,t}$. Since $0 \le t^n_i \le t$ for any $0 \le i \le \gamma_n(t)$, we have $u_n(t^n_i, t, \lambda) \le \overline{c}^u_{t,\lambda}$ and in particular $|\epsilon_{i-1,n}(u_n(t^n_i, t, \lambda))| \le \overline{c}^{\epsilon}_{n,t}(\overline{c}^u_{t,\lambda})$ for all $\gamma_n(s) < i \le \gamma_n(s')$. Using in addition (20) with $C = \overline{c}^u_{t,\lambda}$, we obtain

$$\left|u_n(s,t,\lambda) - u_n(s',t,\lambda)\right| \le \sum_{i=\gamma_n(s)+1}^{\gamma_n(s')} \left(1 + \overline{c}^{\epsilon}_{n,t}(\overline{c}^u_{t,\lambda})\right) c_1(\overline{c}^u_{t,\lambda}) \mu_n(t^n_{i-1}, t^n_i] = \Delta^u_{t,\lambda} \mu_n(s, s']$$

which proves the result. □

For $t, \lambda > 0$, $s \le t$ and $N \ge 1$, define the constants

$$\underline{c}^u_{s,t,\lambda}(N) = \inf\left\{u_n(y,t,\lambda) : s \le y \le t, n \ge N\right\} \text{ and } N_{s,t,\lambda} = \inf\left\{N \ge 1 : \underline{c}^u_{s,t,\lambda}(N) > 0\right\}. \tag{24}$$

The following result shows that u_n is uniformly bounded away from 0 for large enough n.

Lemma 4.5. *Fix $t, \lambda > 0$. Then it holds that*

$$\wp(t) = \sup\left\{s \le t : \liminf_{n\to\infty} \inf_{v\in[s,t]} u_n(v,t,\lambda) = 0\right\}.$$

In particular, the function $t \mapsto \wp(t)$ is increasing and $N_{s,t,\lambda}$ is finite for every $s \in (\wp(t), t]$.

In the sequel, for $t, \lambda > 0$ and $\wp(t) < s \le t$ we note for simplicity $\underline{c}^u_{s,t,\lambda} = \underline{c}^u_{s,t,\lambda}(N_{s,t,\lambda})$ which satisfies $\underline{c}^u_{s,t,\lambda} > 0$.

Proof of Lemma 4.5. For $t, \lambda > 0$ define the two sets

$$\mathscr{S}(t) = \left\{s \le t : \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \inf_{v\in[s,t]} \mathbb{P}\left(X_n(t) > \varepsilon \mid X_n(v) = 1\right) = 0\right\}$$

and

$$\mathscr{S}(t,\lambda) = \left\{s \le t : \liminf_{n\to\infty} \inf_{v\in[s,t]} u_n(v,t,\lambda) = 0\right\}$$

so that $\wp(t) = \sup \mathscr{S}(t)$ and $\wp(t,\lambda) = \sup \mathscr{S}(t,\lambda)$. Fix in the rest of the proof $t, \lambda > 0$. Let $s \le t$: the following statements are equivalent, which proves that $\mathscr{S}(t) = \mathscr{S}(t,\lambda)$ and implies that $\wp(t) = \wp(t,\lambda)$:

(i) $\liminf_{n\to\infty} \inf_{v\in[s,t]} u_n(v,t,\lambda) = 0$;

(ii) there exist sequences $(n(k))$ and (v_k) such that $v_k \in [s,t]$ for each $k \ge 1$ and

$$\lim_{k\to+\infty} n(k) = +\infty \text{ and } \lim_{k\to+\infty} u_{n(k)}(v_k, t, \lambda) = 0;$$

(iii) there exist sequences $(n(k))$ and (v_k) such that $v_k \in [s,t]$ for each $k \ge 1$ and

$$\lim_{k\to+\infty} n(k) = +\infty \text{ and } \lim_{k\to+\infty} \mathbb{E}\left(e^{-\lambda X_{n(k)}(t)} \mid X_{n(k)}(v_k) = 1\right) = 1;$$

(iv) there exist sequences $(n(k))$ and (v_k) such that $v_k \in [s,t]$ for each $k \ge 1$ and for any $\varepsilon > 0$,

$$\lim_{k\to+\infty} n(k) = +\infty \text{ and } \lim_{k\to+\infty} \mathbb{P}\left(X_{n(k)}(t) > \varepsilon \mid X_{n(k)}(v_k) = 1\right) = 0;$$

(v) $\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \inf_{v\in[s,t]} \mathbb{P}\left(X_n(t) > \varepsilon \mid X_n(v) = 1\right) = 0$.

The equivalence between (iii) and (iv) relies on the fact that both conditions are equivalent to the following one: the sequence of random variables $(X_{n(k)}(t), k \ge 1)$ under $\mathbb{P}(\cdot \mid X_{n(k)}(v_k) = 1)$ converges in distribution to 0. Let us also explain the last equivalence. The condition (iv) implies that

$$\liminf_{n\to\infty} \inf_{v\in[s,t]} \mathbb{P}\left(X_n(t) > \varepsilon \mid X_n(v) = 1\right) = 0$$

for every $\varepsilon > 0$, which is stronger than (v). Now, assuming that (v) holds, one can find sequences $(n(k))$, (ε_k) and (v_k) such that $v_k \in [s,t]$ and

$$\lim_{k\to+\infty} n(k) = +\infty, \lim_{k\to+\infty} \varepsilon_k = 0 \text{ and } \lim_{k\to+\infty} \mathbb{P}\left(X_{n(k)}(t) > \varepsilon_k \mid X_{n(k)}(v_k) = 1\right) = 0.$$

Then the sequences $(n(k))$ and (v_k) satisfy (iv) since for any $\varepsilon > 0$,

$$\mathbb{P}\left(X_{n(k)}(t) > \varepsilon \mid X_{n(k)}(v_k) = 1\right) \le \mathbb{P}\left(X_{n(k)}(t) > \varepsilon_k \mid X_{n(k)}(v_k) = 1\right)$$

for k large enough, since $\varepsilon_k \to 0$.

We now prove that $\wp(\cdot)$ is an increasing function. Let $t' > t$: we will show that $\mathscr{S}(t, \overline{c}^u_{t',\lambda}) \subset \mathscr{S}(t',\lambda)$, which proves that $\wp(t) \le \wp(t')$. So consider $s \in \mathscr{S}(t, \overline{c}^u_{t',\lambda})$, i.e., $s \le t$ with $\liminf_{n\to+\infty} \inf_{v\in[s,t]} u_n(s,t,\overline{c}^u_{t',\lambda}) = 0$. Then $s \le t'$, and the composition rule (1) together with the monotonicity of u_n in λ give

$$u_n(s,t',\lambda) = u_n(s,t,u_n(t,t',\lambda)) \le u_n(s,t,\overline{c}^u_{t',\lambda})$$

which entails

$$\liminf_{n\to+\infty} \inf_{v\in[s,t']} u_n(s,t',\lambda) \le \liminf_{n\to+\infty} \inf_{v\in[s,t]} u_n(s,t,\overline{c}^u_{t',\lambda}).$$

Since this last quantity is equal to 0 this proves that $s \in \mathscr{S}(t',\lambda)$ and gives the result.

Finally, the fact that $N_{s,t,\lambda}$ is finite when $\wp(t) < s \le t$ follows readily from the fact that

$$\lim_{N\to+\infty} \underline{c}^u_{s,t,\lambda}(N) = \liminf_{n\to\infty} \inf_{v\in[s,t]} u_n(v,t,\lambda).$$

The result is proved. □

4.4. **Proof of Theorem 2.1.** We now use the results of Sections 4.2 and 4.3 to prove Theorem 2.1. The main idea is to use Gronwall type argument. We will use Gronwall's lemma in the backwards form of Lemma 4.6, while Lemma 4.7 establishes a sort of Lipschitz property of Ψ needed to use Gronwall's lemma. In a similar vein as the following lemma, we refer to Lemma 3.2 in Dynkin [12] which states and proves a particular case of this result to construct superprocesses.

Lemma 4.6 (Backwards Gronwall's lemma)**.** *Let u and R be non-negative, càdlàg functions and let π be a locally finite measure. If*

$$u(s) \le R(s) + \int_{(s,t]} u(x)\pi(dx)$$

holds for all $0 \le s \le t$, then for all $0 \le s \le t$ we have

$$u(s) \le R(s) + e^{\pi(s,t]} \int_{(s,t]} R(x)\pi(x).$$

Proof. It follows the proof of Dynkin. By induction

$$
\begin{aligned}
u(s) &\leq R(s) + \int_{(s,t]} \left[R(s_1) + \int_{(s_1,t]} u(s_2)\pi(ds_2) \right] \pi(ds_1) \\
&\leq \cdots \\
&\leq R(s) + \sum_{i=1}^{n} \int\int \cdots \int 1_{s<s_1<s_2<\cdots<s_i\leq t} R(s_i)\pi(ds_1)\cdots\pi(ds_i) \\
&\leq R(s) + \sum_{i=1}^{n} \int_{(s,t]} R(s_i) \frac{\mu(s_i,t]^{i-1}}{(i-1)!}\pi(ds_i) \\
&\leq R(s) + \sum_{i=1}^{n} \int_{(s,t]} A(s_i) \frac{\pi(s,t]^{i-1}}{(i-1)!}\pi(ds_i) \\
&\leq R(s) + \exp(\mu(s,t]) \int_{(s,t]} A(x)\pi(dx) + \epsilon_n(s,t,\lambda)
\end{aligned}
$$

where $\epsilon_n(s,t)$ yields the rest of the Taylor expansion of the exponential function :

$$
\epsilon_n(s,t) \leq C \int_{(s,t]} R(x)\pi(dx).\frac{\mu(s,t]^n}{n!} \overset{n\to\infty}{\longrightarrow} 0.
$$

This completes the proof. □

In the rest of this section, we assume that the two assumptions (A1) and (A2) of Theorem 2.1 hold and we consider the measures α, β and ν given there. Recall that $\mu_n = |\alpha_n| + \beta_n$, and define analogously $\mu = |\alpha| + \beta$, in particular we have

$$
\lim_{n\to+\infty} \mu_n(t) = \mu(t). \tag{25}
$$

Define also the measure $\widetilde{\mu}$ by

$$
\widetilde{\mu}(A) = \mu(A) + \int_{(0,\infty)\times A} \frac{x^2}{1+x^2}\nu(dx\,dy), \; A \in \mathscr{B}.
$$

For $0 < \eta < T$ define the constants $c_2(\eta,T)$ and $c_3(\eta,T)$ as follows:

$$
c_2(\eta,T) = \sup_{\substack{\eta\leq y,y'\leq T \\ 0\leq x}} \left| \frac{h(x,y)-h(x,y')}{(y-y')x^2/(1+x^2)} \right| \text{ and } c_3(\eta,T) = 1 + T + c_2(\eta,T). \tag{26}
$$

Note that these constants are monotone in η and T.

Lemma 4.7. *For any $0 < \eta < T$, the constants $c_2(\eta,T)$ and $c_3(\eta,T)$ are finite. Moreover, for any $A \in \mathscr{B}$ we have*

$$
\int_{(0,\infty)\times A} \frac{x^2}{2(1+x^2)}\nu(dx\,dy) \leq \beta(A) \tag{27}
$$

and in particular the measure $\widetilde{\mu}$ is σ-finite. Finally, for any measurable, positive functions f_1 and f_2 and any $A \in \mathscr{B}$, we have

$$
\left|\Psi(f_1)(A) - \Psi(f_2)(A)\right| \leq c_3\left(\inf_A f_1 \wedge \inf_A f_2, \sup_A f_1 + \sup_A f_2\right) \int_A |f_1 - f_2| d\widetilde{\mu}.
$$

Proof. Let $0 < \eta < T$ and $\eta \leq y, y' \leq T$, and fix $x \geq 0$: the constant $c_2(\eta,T)$ is finite because

$$
\left| \frac{h(x,y)-h(x,y')}{(y-y')x^2/(1+x^2)} \right| \leq \sup_{\eta\leq v\leq T} \left|H'(v)\right|,
$$

with $H(y) = h(x,y)(1+x^2)/x^2$. One can compute $H'(y) = xe^{-yx} + y\Phi_1(yx)$ and so

$$\sup_{\eta \le v \le T} \left|H'(v)\right| \le \frac{1}{\eta} \sup_{v\ge 0}(ve^{-v}) + T \sup_{v \ge 0} |\Phi_1(v)|.$$

This upper bound being independent of x, we get the finiteness of $c_2(\eta, T)$, and hence of $c_3(\eta, T)$. As for (27), we have

$$\begin{aligned}
\int_{(0,\infty)\times A} \frac{x^2}{2(1+x^2)} \nu(dx\,dy) &\overset{\text{(i)}}{=} \int_{(0,\infty)} \frac{x}{(1+x^2)^2} \nu([x,\infty)\times A)\,dx \\
&\overset{\text{(ii)}}{=} \int_{(0,\infty)} \frac{x}{(1+x^2)^2} \liminf_{n\to+\infty} \nu_n([x,\infty)\times A)\,dx \\
&\overset{\text{(iii)}}{\le} \liminf_{n\to+\infty} \int_{(0,\infty)} \frac{x}{(1+x^2)^2} \nu_n([x,\infty)\times A)\,dx \\
&\overset{\text{(iv)}}{=} \liminf_{n\to+\infty} \int_{(0,\infty)\times A} \frac{x^2}{2(1+x^2)} \nu_n(dx\,dy) \overset{\text{(v)}}{\le} \liminf_{n\to+\infty} \big(\beta_n(A)\big)
\end{aligned}$$

using Fubini's theorem for (i) and (iv), the assumption (A1) for (ii) (using also that the set $\{x : \nu(\{x\}\times A) > 0\}$ has zero Lebesgue measure), Fatou's lemma for (iii) and finally the definition of ν_n and β_n for (v). Since $\mu = |\alpha| + \beta$ this implies the σ-finiteness of $\widetilde{\mu}$. Consider finally f_1 and f_2 two measurable, positive functions, $A \in \mathscr{B}$ and let $\eta_A = \inf_A f_1 \wedge \inf_A f_2$ and $T_A = \sup_A f_1 + \sup_A f_2$: then by definition (14) of Ψ we have

$$\begin{aligned}
\left|\Psi(f_1)(A) - \Psi(f_2)(A)\right| \le \int_A |f_1 - f_2| d|\alpha| + \int_A |f_1^2 - f_2^2| d\beta \\
+ \int_{(0,\infty)\times A} \left|h(x, f_1(y)) - h(x, f_2(y))\right| \nu(dx\,dy).
\end{aligned}$$

Using $|f_1^2 - f_2^2| = |f_1 - f_2|(f_1 + f_2)$ and plugging in the constant c_2, we obtain

$$\begin{aligned}
\left|\Psi(f_1)(A) - \Psi(f_2)(A)\right| \le \int_A |f_1 - f_2| d|\alpha| + T_A \int_A |f_1 - f_2| d\beta \\
+ c_2(\eta_A, T_A) \int_{(0,\infty)\times A} \left|f_1(y) - f_2(y)\right| \frac{x^2}{1+x^2} \nu(dx\,dy) \le c_3(\eta_A, T_A) \int_A |f_1 - f_2| d\widetilde{\mu}
\end{aligned}$$

with $T_A = \sup_A f_1 + \sup_A f_2$ and $\eta_A = \inf_A f_1 \wedge \inf_A f_2$, which was to be proved. □

Before finally turning to the proof of Theorem 2.1, we state an intermediate result whose long and tedious proof is postponed to the appendix.

Lemma 4.8. *Fix $t, \lambda > 0$ and consider any sequence (ℓ_n) with $\ell_n \to \lambda$. For $n \ge 1$, let R_n be the function*

$$R_n(s) = |\Psi_n(u_n(\cdot, t, \ell_n))((s,t]) - \Psi(u_n(\cdot, t, \ell_n))((s,t])|, \ 0 \le s \le t.$$

Then $R_n(s) \to 0$ for any $\wp(t) < s \le t$ and $\sup\{R_n(s) : 0 \le s \le t, n \ge 1\}$ is finite.

Theorem 2.1 is now a direct consequence of Lemmas 4.9 and 4.10.

Lemma 4.9. *Fix $t, \lambda > 0$ and a sequence (ℓ_n) with $\ell_n \to \lambda$. Then for any $s \in [\wp(t), t]$, the sequence $(u_n(s,t,\ell_n), n \ge 1)$ converges and the function*

$$u : s \in [\wp(t), t] \mapsto \lim_{n\to+\infty} u_n(s, t, \ell_n)$$

is the unique function satisfying the following properties:

(1) $u(s) = \lambda + \Psi(u)((s,t])$ for all $\wp(t) \le s \le t$;
(2) u is càdlàg;

(3) $\inf_{[s,t]} u > 0$ *for any* $\wp(t) < s \le t$.

From this lemma, one sees in particular that for any $s \in [\wp(t), t]$, the limit of the sequence $(u_n(s,t,\ell_n), n \ge 1)$ depends on (ℓ_n) only through its limit, i.e., if $(\ell'_n, n \ge 1)$ is another sequence with $\ell'_n \to \lambda$ then $\lim_{n\to+\infty} u_n(s,t,\ell_n) = \lim_{n\to+\infty} u_n(s,t,\ell'_n)$.

Proof of Lemma 4.9. In the rest of the proof fix $t, \lambda > 0$ and (ℓ_n) a sequence converging to λ. Let $\ell = \inf_{n\ge1} \ell_n$ and $L = \sup_{n\ge1} \ell_n$ and assume without loss of generality, since $\ell_n \to \lambda > 0$, that $\ell > 0$. To ease the notation, note in the rest of the proof $\wp = \wp(t)$ and $u_n(s) = u_n(s,t,\ell_n)$ for $0 \le s \le t$. We decompose the proof in four steps: first we prove that the sequence $(u_n(s), n \ge 1)$ is Cauchy for any $s \in (\wp, t]$, then that it is Cauchy for $s = \wp$, then that u satisfies the claimed properties and finally that it is the only such function.

Before beginning, note that everything is trivial if $\wp = t$, because then $u_n(s) = \ell_n$ and $\Psi(u)((s,t]) = 0$ for any $s \in [\wp, t]$. Hence in the sequel we assume that $\wp < t$.

First step: $(u_n(s))$ *is Cauchy for* $s \in (\wp, t]$. In the rest of this step fix $s \in (\wp, t]$ and for $s \le y \le t$ define $R_n(y) = |\Psi_n(u_n)((y,t]) - \Psi(u_n)((y,t])|$. Then (17) gives for any $s \le y \le t$ and any $m, n \ge 1$

$$\left|u_n(y) - u_m(y)\right| \le R_n(y) + R_m(y) + \left|\Psi(u_n)((y,t]) - \Psi(u_m)((y,t])\right|.$$

Lemma 4.7 gives

$$\left|\Psi(u_n)((y,t]) - \Psi(u_m)((y,t])\right| \le c_3\left(\inf_{(y,t]} u_n \wedge \inf_{(y,t]} u_m, \sup_{(y,t]} u_n + \sup_{(y,t]} u_m\right)\int_{(y,t]} |u_n - u_m|\, d\widetilde{\mu}.$$

Since the function $u_n(s,t,\lambda)$ is increasing in λ, we have for any $y \in [s,t]$ and $n \ge N_{s,t,\ell}$ (recall that $N_{s,t,\ell}$ is defined in (24) and is finite by Lemma 4.5)

$$u_n(y) = u_n(y,t,\ell_n) \ge u_n(y,t,\ell) \ge \inf_{v\in[s,t]} u_n(v,t,\ell) \ge \underline{c}^u_{s,t,\ell} > 0.$$

Similar monotonicity arguments lead to $u_n(y) \le \overline{c}^u_{t,L}$ for any $y \le t$ and $n \ge 1$, so that monotonicity properties of $c_3(\eta, T)$ in η and T give for $n, m \ge N_{s,t,\lambda}$

$$\left|\Psi(u_n)((y,t]) - \Psi(u_m)((y,t])\right| \le c_3\left(\underline{c}^u_{s,t,\ell}, 2\overline{c}^u_{t,L}\right)\int_{(y,t]} |u_n - u_m|\, d\widetilde{\mu}.$$

We finally get the bound

$$\left|u_n(y) - u_m(y)\right| \le R_n(y) + R_m(y) + C\int_{(y,t]} |u_n - u_m|\, d\widetilde{\mu}$$

with $C = C_{s,t,\ell,L} = c_3(\underline{c}^u_{s,t,\ell}, 2\overline{c}^u_{t,L})$, which holds for all $s \le y \le t$ and all $n, m \ge N_{s,t,\ell}$. Thus Lemma 4.6 implies for those n, m

$$|u_n(s) - u_m(s)| \le R_n(s) + R_m(s) + Ce^{C\widetilde{\mu}(s,t]}\int_{(s,t]} (R_n + R_m)\, d\widetilde{\mu}$$

so that for any $n_0 \ge N_{s,t,\ell}$,

$$\sup_{n,m\ge n_0} |u_n(s) - u_m(s)| \le 2\sup_{n\ge n_0} (R_n(s)) + 2Ce^{C\widetilde{\mu}(s,t]}\sup_{n\ge n_0}\left(\int_{(s,t]} R_n d\widetilde{\mu}\right).$$

Lemma 4.8 combined with the dominated convergence theorem shows that the right hand side of the above inequality goes to 0 as $n_0 \to +\infty$ which proves that the sequence $(u_n(s), n \ge 1)$ is Cauchy and completes the proof of this first step.

Second step: $(u_n(\wp))$ *is Cauchy.* For any $\wp < s' \leq t$, Lemma 4.4 entails

$$\begin{aligned}\left|u_n(\wp) - u_m(\wp)\right| &\leq \left|u_n(\wp) - u_n(s')\right| + \left|u_m(\wp) - u_m(s')\right| + \left|u_n(s') - u_m(s')\right|\\ &\leq 2\Delta^u_{t,\ell_n}\mu_n(\wp, s'] + \left|u_n(s') - u_m(s')\right|\\ &\leq 2\Delta^u_{t,L}\mu_n(\wp, s'] + \left|u_n(s') - u_m(s')\right|\end{aligned}$$

using for the last inequality that $\ell_n \leq L$ and that $\Delta^u_{t,y}$ is increasing in y, as can be seen directly from its definition (23). Hence for any $n_0 \geq 1$,

$$\sup_{m,n\geq n_0} \left|u_n(\wp) - u_m(\wp)\right| \leq 2\Delta^u_{t,L} \sup_{n\geq n_0} \mu_n(\wp, s'] + \sup_{m,n\geq n_0} \left|u_n(s') - u_m(s')\right|.$$

By (25) and the fact that $(u_n(s'))$ is Cauchy by the first step since $\wp < s' \leq t$, the right hand side of the above inequality goes to $2\Delta^u_{t,L}\mu(\wp, s']$ as n_0 goes to infinity. Since $\mu(\wp, s'] \to 0$ as $s' \downarrow \wp$, letting $s' \downarrow \wp$ shows that $(u_n(\wp))$ is Cauchy.

Third step: properties of u. Let from now on u denote the function of the statement and consider $s \in [\wp, t]$. First note that the second property follows readily from the first one, so we only have to prove the first and third ones. Assume first that $s > \wp$. We have seen in the first step that for any $s \leq y \leq t$ and $n \geq N_{s,t,\ell}$

$$0 < \underline{c}^u_{s,t,\ell} \leq u_n(y) \leq \overline{c}^u_{t,L} < +\infty.$$

Since $u_n(y) \to u(y)$ for $s \leq y \leq t$ by definition of u, u also satisfies $\underline{c}^u_{s,t,\lambda} \leq u(y) \leq \overline{c}^u_{t,L}$ for $s \leq y \leq t$. In particular the third property $\inf_{[s,t]} u > 0$ is satisfied. Let us now show the first property, still in the case $s > \wp$. Plugging in (17), we get

$$\begin{aligned}|u(s) - \lambda - \Psi(u)((s,t])| \leq |u(s) - u_n(s)| + |\Psi_n(u_n)((s,t]) - \Psi(u_n)((s,t])|\\ + |\Psi(u_n)((s,t]) - \Psi(u)((s,t])|.\end{aligned}$$

Since both u_n and u are bounded uniformly on $[s,t]$ by $\underline{c}^u_{s,t,\ell}$ and $\overline{c}^u_{t,L}$, Lemma 4.7 gives with similar arguments as in the first step

$$|\Psi(u_n)((s,t]) - \Psi(u)((s,t])| \leq c_3\left(\underline{c}^u_{s,t,\ell}, 2\overline{c}^u_{t,L}\right)\int_{(s,t]} |u_n - u| d\widetilde{\mu}$$

and finally, we have for $n \geq N_{s,t,\ell}$

$$\begin{aligned}|u(s) - \lambda - \Psi(u)((s,t])| \leq |u(s) - u_n(s)| + |\Psi_n(u_n)((s,t]) - \Psi(u_n)((s,t])|\\ + c_3\left(\underline{c}^u_{s,t,\ell}, 2\overline{c}^u_{t,L}\right)\int_{(s,t]} |u_n - u| d\widetilde{\mu}.\end{aligned}$$

Let now n go to infinity. The first term of the above upper bound goes to 0 by definition of $u(s)$; the second term goes to 0 by Lemma 4.8. Finally, the last term also goes to 0 using the dominated convergence theorem. Thus u satisfies the first property for $s > \wp$.

To extend this for $s = \wp$, we proceed as in the second step and consider any $\wp < s' \leq t$: then $|u_n(\wp) - u_n(s')| \leq \Delta^u_{t,L}\mu_n(\wp, s']$ and taking the limit $n \to +\infty$ gives $|u(\wp) - u(s')| \leq \Delta^u_{t,L}\mu(\wp, s']$. Letting $s' \downarrow \wp$ shows that $u(s') \to u(\wp)$. On the other hand, it is plain that $\lambda + \Psi(u)((s', t]) \to \lambda + \Psi(u)((\wp, t])$ and so u satisfies the first property for all $s \in [\wp, t]$. It remains to show uniqueness in order to complete the proof.

Fourth step: uniqueness. Now let us prove uniqueness: let $\tilde{u}$ be a function with the same properties. Then Lemma 4.7 gives

$$|u(s)-\tilde{u}(s)| = |\Psi(u)((s,t])-\Psi(\tilde{u})((s,t])| \le c_3\left(\underline{c}_{s,t,\ell}\wedge \inf_{[s,t]}\tilde{u},\, \overline{c}^u_{t,L}+\sup_{[s,t]}\tilde{u}\right)\int_{(s,t]}|u-\tilde{u}|d\mu$$

and we conclude that $u=\tilde{u}$ using Lemma 4.6 (note that $\sup_{[s,t]}\tilde{u}$ is finite because $\tilde{u}$ is càdlàg). □

Lemma 4.10. *For any $t\ge 0$, $\alpha\{t\}\ge -1$,*

$$\beta\{t\} = \int_{(0,\infty)\times\{t\}}\frac{x^2}{2(1+x^2)}\nu(dx\,dy) \;\; \text{and} \;\; \int_{(0,\infty)\times(0,t]}\left(1\wedge x^2\right)\nu(dx\,dy) < +\infty.$$

Proof. The first assertion is a trivial consequence of Assumption (A2) since $\alpha_{i,n}\ge -1$ for every $i\ge 0$ and $n\ge 1$ and the last assertion on $\int_{(0,\infty)\times(0,t]}\left(1\wedge x^2\right)\nu(dx\,dy)$ readily follows from (27). Let us now prove the result on $\beta\{t\}$. First, note that the equality must hold by (27) when $\beta\{t\}=0$, so assume in the sequel that $\beta\{t\}>0$. In particular, the assumption (A2) implies that $\beta_n\{t\}\to\beta\{t\}$.

For every $x>0$ such that $\nu(\{x\}\times\{t\})=0$, $\nu_{\gamma_n(t),n}[x,\infty)$ converges to $\nu([x,\infty)\times\{t\})$ by Assumption (A2). Then for any $d>0$ with $\nu(\{d\}\times\{t\})=0$, we get by weak convergence of probability measures (since all the measures restricted to $[d,\infty)$ have finite mass) and the dominated convergence theorem

$$\lim_{n\to+\infty}\int_{(d,\infty)}\frac{x^2}{2(1+x^2)}\nu_{\gamma_n(t),n}(dx) = \int_{(d,\infty)\times\{t\}}\frac{x^2}{2(1+x^2)}\nu(dx\,dy). \tag{28}$$

On the other hand, we have

$$\int_{[-1/n,d]}\frac{x^2}{1+x^2}\nu_{\gamma_n(t),n}(dx) \le \left(d+\frac{1}{n}\right)\int_{[-1/n,d]}\frac{|x|}{1+x^2}\nu_{\gamma_n(t),n}(dx)$$

and from the definition of $\alpha_{\gamma_n(t),n}$ we see that

$$\begin{aligned}\int_{[-1/n,d]}\frac{|x|}{1+x^2}\nu_{\gamma_n(t),n}(dx) &= \int_{[-1/n,d]}\frac{x}{1+x^2}\nu_{\gamma_n(t),n}(dx) + \frac{2/n}{1+(1/n)^2}\nu_{\gamma_n(t),n}\{-1/n\}\\ &\le \alpha_{\gamma_n(t),n}+\frac{2n^2}{1+n^2}\end{aligned}$$

using that $\nu_{\gamma_n(t),n}\{-1/n\}\le\nu_{\gamma_n(t),n}(\mathbb{R})=n$ for the last inequality. Since the sequence $(\alpha_{\gamma_n(t),n})$ is bounded (since $|\alpha_{\gamma_n(t),n}|\le|\alpha_n|(t+1)$ for n large enough and $|\alpha|(t+1)\to|\alpha|(t+1)$), say by some constant C, we obtain

$$\limsup_{n\to+\infty}\int_{[-1/n,d]}\frac{x^2}{1+x^2}\nu_{\gamma_n(t),n}(dx)\le Cd.$$

Letting $d\to 0$ gives

$$\lim_{d\to 0}\limsup_{n\to+\infty}\int_{[-1/n,d]}\frac{x^2}{1+x^2}\nu_{\gamma_n(t),n}(dx)=0$$

which combined with (28) gives

$$\begin{aligned}\lim_{n\to+\infty}\int_{[-1/n,\infty)}\frac{x^2}{2(1+x^2)}\nu_{\gamma_n(t),n}(dx) &= \lim_{d\to 0}\int_{(d,\infty)\times\{t\}}\frac{x^2}{2(1+x^2)}\nu(dx\,dy)\\ &= \int_{(0,\infty)\times\{t\}}\frac{x^2}{2(1+x^2)}\nu(dx\,dy).\end{aligned}$$

Since $2\beta_n\{t\}=\int x^2/(1+x^2)\nu_{\gamma_n(t),n}(dx)$ and $\beta_n\{t\}\to\beta\{t\}$ this proves the result. □

4.5. **Proof of Corollary 2.3.** We now prove Corollary 2.3, so let $t \geq 0$, $\wp(t) \leq s \leq t$, $I \geq 1$, $\lambda_i > 0$ for $i = 1, \dots, I$ and $s \leq t_1 < \cdots < t_I \leq t$. For simplicity we treat the case $I = 2$, the general case following similarly but with more computation. We must show that

$$\lim_{n \to +\infty} \mathbb{E}\left(\exp\left(-\lambda_1 X_n(t_1) - \lambda_2 X_n(t_2)\right) \mid X_n(s) = 1\right) = \exp\left(-u(s, t_1, \lambda_1 + u(t_1, t_2, \lambda_2))\right).$$

Since $\wp$ is an increasing function by Lemma 4.5 and $\wp(t) < s \leq t_1 < t_2 \leq t$ we get

$$\wp(s) \leq \wp(t_1) \leq \wp(t_2) \leq \wp(t) < s \leq t_1 < t_2 \leq t.$$

In particular, $\wp(t_2) < t_1 \leq t_2$ so Lemma 4.9 implies that $u_n(t_1, t_2, \lambda_2) \to u(t_1, t_2, \lambda_2)$. Also, $\wp(t_1) < s \leq t_1$ so Lemma 4.9 implies that

$$\lim_{n \to +\infty} u_n(s, t_1, \lambda_1 + u_n(t_1, t_2, \lambda_2)) = u(s, t_1, \lambda_1 + u(t_1, t_2, \lambda_2))$$

which proves the desired result using (18).

5. Proof of Corollaries 2.4 and 2.5

5.1. **Proof of Corollary 2.4.** We first show that (4) implies that $\wp(t) = 0$. Remember the definition of the set $\mathscr{S}(t)$ introduced in the proof of Lemma 4.5:

$$\mathscr{S}(t) = \left\{ s \leq t : \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \inf_{v \in [s,t]} \mathbb{P}\left(X_n(t) > \varepsilon \;\middle|\; X_n(v) = 1\right) = 0 \right\}$$

so that $\wp(t) = \sup \mathscr{S}(t)$. By monotonicity, either $\mathscr{S}(t)$ is empty or it is an interval containing 0. The condition (4) implies directly that $0 \notin \mathscr{S}(t)$, so that $\mathscr{S}(t) = \emptyset$ and by convention $\wp(t) = 0$. In the rest of this subsection we show that the two other conditions of Corollary 2.4 imply (4). Note that under the assumptions of the following lemma, Lemma 4.3 ensures that $\bar{c}^u_{t,\lambda} < +\infty$ and so $\min(1/\bar{c}^u_{t,\lambda}, 1) > 0$.

Lemma 5.1. *Fix $t, \lambda > 0$ and assume that the two sequences $(|\alpha_n|(t))$ and $(\beta_n(t))$ are bounded. Fix some $0 < a \leq \min(1/\bar{c}^u_{t,\lambda}, 1)$ and for $n \geq 1$ and $0 \leq i < \gamma_n(t)$ define the two following quantities:*

$$\widehat{\alpha}_{i,n} := 1 + \left(1 + \epsilon_{i,n}(u_n(t^n_{i+1}, t, \lambda))\right) \int_{[-1/n, a]} x \nu_{i,n}(dx)$$

and

$$\widehat{\beta}_{i,n} := \left(1 + \epsilon_{i,n}(u_n(t^n_{i+1}, t, \lambda)\right) \int_{[-1/n, a]} x^2 \nu_{i,n}(dx).$$

Then there exists $n_0 = n_0(t, \lambda, a)$ such that for all $n \geq n_0$ and all $0 \leq i < \gamma_n(t)$,

$$u_n(t^n_i, t, \lambda) \geq \widehat{\alpha}_{i,n} u_n(t^n_{i+1}, t, \lambda) - \widehat{\beta}_{i,n} (u_n(t^n_{i+1}, t, \lambda))^2$$

and also

$$\widehat{\alpha}_{i,n} \geq 1 - 2\left(|\alpha_{i,n}| + a^{-1} \beta_{i,n} + \mu_{i,n}\{0\}/n^2\right), \; \widehat{\beta}_{i,n} \leq 4 \min(a(1 + \alpha_{i,n}), \beta_{i,n}).$$

Proof. By Lemma 4.2 let $n_{t,a} \geq 1$ be such that $1 + \epsilon_{i,n}(y) \in [0, 2]$ for all $n \geq n_{t,\lambda}$, $y \leq 1/a$ and $0 \leq i < \gamma_n(t)$. By definition we have $\psi_{i,n}(y) = \left(1 + \epsilon_{i,n}(y)\right) \int (1 - e^{-yx}) \nu_{i,n}(dx)$ and since $1 - e^{-x} \geq x - x^2$ for all $x \geq -1$ we obtain for every $n \geq n_{t,a}$, $y \leq 1/a$ and $0 \leq i < \gamma_n(t)$

$$\psi_{i,n}(y) \geq \left(1 + \epsilon_{i,n}(y)\right) \left(y \int_{[-1/n, a]} x \nu_{i,n}(dx) - y^2 \int_{[0,a]} x^2 \nu_{i,n}(dx) \right).$$

This yields the first inequality of the lemma using the equality

$$u_n(t^n_i, t, \lambda) = u_n(t^n_{i+1}, t, \lambda) + \psi_{i,n}(u_n(t^n_{i+1}, t, \lambda))$$

that stems from Lemma 4.1 and the fact that $u_n(t^n_{i+1}, t, \lambda) \le \overline{c}^u_{t,\lambda} \le 1/a$ for all $n \ge 1$ and $i < \gamma_n(t)$. Moreover,

$$\begin{aligned}
\int_{[-1/n,a]} x\nu_{i,n}(dx) &\ge (1+1/n^2)\int_{[-1/n,0]} \frac{x}{1+x^2}\nu_{i,n}(dx) + \int_{[0,a]} \frac{x}{1+x^2}\nu_{i,n}(dx) \\
&\ge \int_{[-1/n,a]} \frac{x}{1+x^2}\nu_{i,n}(dx) - \mu_{i,n}\{0\}/n^2 \\
&\ge \int_{[-1/n,\infty)} \frac{x}{1+x^2}\nu_{i,n}(dx) - a^{-1}\int_{(a,\infty)} \frac{x^2}{1+x^2}\nu_{i,n}(dx) - \mu_{i,n}\{0\}/n^2 \\
&\ge \alpha_{i,n} - a^{-1}\beta_{i,n} - \mu_{i,n}\{0\}/n^2.
\end{aligned}$$

This gives the second inequality of the lemma whereas the last one comes from $1+x^2 \le 2$ if $-1/n \le x \le a$, which gives

$$\begin{aligned}
\int_{[-1/n,a]} x^2\nu_{i,n}(dx) &\le 2\max(a,1/n)\int_{[-1/n,a]} \frac{|x|}{1+x^2}\nu_{i,n}(dx) \\
&\le 2\max(a,1/n)\left(1+\int_{[-1/n,a]} \frac{x}{1+x^2}\nu_{i,n}(dx)\right) \le 2a(1+\alpha_{i,n}),
\end{aligned}$$

for n large enough and every $i \le \gamma_n(t)$. Together with $\epsilon_{i,n}(u_n(t^n_{i+1}, t, \lambda)) \in [0,2]$ this completes the proof. □

Lemma 5.2. *Let $(b_i, i \ge 0)$ and $(c_i, i \ge 0)$ be two sequences of respectively positive and non-negative real numbers such that there exist $\epsilon, M > 0$ such that $b_i^2 - c_i b_i M \ge \epsilon$ for every $i \ge 0$. If a non-negative, finite sequence $(w_i, 0 \le i \le I)$ satisfies $w_i \ge w_{i+1}b_i - w_i^2 c_i$ and $w_i \le M$ for every $0 \le i \le I$, then for every $0 \le i \le I$,*

$$w_i \ge \left(\frac{1}{w_I}\Gamma_i^{I-1} + \sum_{k=i}^{I-1}\Gamma_i^{k-1}\delta_k\right)^{-1}$$

where $\delta_k = c_k/b_k^2$, $\Gamma_i^{i-1} = 1$ and $\Gamma_j^i = \prod_{k=j}^i \gamma_k$ for $j \le i$, with $\gamma_k = (1 + c_k M^2/\epsilon)/b_k$.

Proof. In the rest of the proof note $\rho_k = c_k M^2/\epsilon$ and for $x \le M$ let

$$r_i(x) = c_i^2 \frac{x^2}{b_i^2 - b_i c_i x}.$$

Since by assumption $b_i^2 - b_i c_i M \ge \epsilon$ we have $b_i^2 - b_i c_i x \ge \epsilon$ for all $x \le M$ and so r_i is increasing and positive on $(-\infty, M]$. In particular, we have $r_i(x) \le r_i(M)$ and since by assumption $r_i(M) \le c_i^2 M^2/\epsilon$ we obtain $r_i(x) \le \rho_i$ for all $i \ge 1$ and $x \le M$. Moreover, one can check that r_i satisfies

$$b_i x - c_i x^2 = b_i x\left(1 + r_i(x) + \frac{c_i}{b_i}x\right)^{-1}.$$

The left hand side corresponds to a Taylor expansion with rest $r_i(x)$. We can compose it recursively thanks to the stability of homographies. This kind of techniques is used in the study of branching processes in random environment with linear fractional offspring distribution. Since $r_i(x) \le \rho_i$ we obtain from the previous equation for any $x \le M$

$$b_i x - c_i x^2 \ge b_i x\left(1 + \rho_i + \frac{c_i}{b_i}x\right)^{-1} = b_i\left(\frac{1+\rho_i}{x} + \frac{c_i}{b_i}\right)^{-1}.$$

In particular, since by assumption $w_{i+1} \le M$ and $w_i \ge w_{i+1} b_i - w_i^2 c_i$ we obtain

$$w_i \ge w_{i+1} b_i - w_i^2 c_i \ge b_i \left(\frac{1+\rho_i}{w_{i+1}} + \frac{c_i}{b_i} \right)^{-1}.$$

By backwards induction one immediately sees that $w_i \ge v_{I,i}$ for all $0 \le i \le I$, where $v_{I,I} = w_I$ and

$$v_{I,i} = b_i \left(\frac{1+\rho_i}{v_{I,i+1}} + \frac{c_i}{b_i} \right)^{-1}, \ i \ge 0.$$

The definition of $v_{I,i}$ gives rise to the backwards recursion $\overline{v}_{I,i} = \gamma_i \overline{v}_{I,i+1} + \delta_i$ with $\overline{v}_i = 1/v_i$, from which one easily deduces that $\overline{v}_{I,i} = \overline{v}_{I,I} \Gamma_i^{I-1} + \sum_{k=i}^{I-1} \Gamma_i^{k-1} \delta_k$. This concludes the proof. □

We can now combine the two previous lemmas to obtain the following result which yields the Corollary.

Lemma 5.3. *Fix $t, \lambda > 0$ and assume that*

$$C_t = \sup_{n \ge 1} \left(|\alpha_n|(t) + \beta_n(t) + n^{-2} \sum_{i=0}^{\gamma_n(t)} \mu_{i,n}\{0\} \right)$$

is finite. Assume also that there exists $\epsilon \in (0,1)$ and $a_0 > 0$ such that

$$\sum_{k=0}^{an} k \mu_{i,n}\{k\} \ge \epsilon$$

for all $n \ge 1$, $a < a_0$ and $0 \le i < \gamma_n(t)$. Then (4) holds.

Proof. Fix in the rest of the proof t and $\lambda > 0$ and let C_t be as in the statement of the lemma. Let $a_{t,\lambda} > 0$ be defined as follows:

$$a_{t,\lambda} = \min \left(\frac{a_0}{2}, \frac{1}{\overline{c}^u_{t,\lambda}}, 1, \frac{\epsilon}{16 \overline{c}^u_{t,\lambda}(1+C_t)}, \frac{C_t}{1+C_t} \right).$$

Remember the quantities $\widehat{\alpha}_{i,n}$ and $\widehat{\beta}_{i,n}$ defined in Lemma 5.1. Since by definition $a_{t,\lambda} \le \min(1/\overline{c}^u_{t,\lambda}, 1)$, this lemma guarantees the existence of $n_{t,\lambda}$ such that for all $n \ge n_{t,\lambda}$ and $0 \le i < \gamma_n(t)$,

$$u_n(t_i^n, t, \lambda) \ge \widehat{\alpha}_{i,n} u_n(t_{i+1}^n, t, \lambda) - \widehat{\beta}_{i,n} (u_n(t_{i+1}^n, t, \lambda))^2$$

and also

$$\widehat{\alpha}_{i,n} \ge 1 - 2\left(|\alpha_{i,n}| + a_{t,\lambda}^{-1} \beta_{i,n} + \mu_{i,n}\{0\}/n^2 \right), \ \widehat{\beta}_{i,n} \le 4 \min\left(a_{t,\lambda}(1+\alpha_{i,n}), \beta_{i,n} \right).$$

By definition of C_t we have $\alpha_{i,n}, \beta_{i,n} \le C_t$ for any $i < \gamma_n(t)$ and so

$$4 \min\left(a_{t,\lambda}(1+\alpha_{i,n}), \beta_{i,n} \right) \le 4 \min\left(a_{t,\lambda}(1+C_t), C_t \right) = 4 a_{t,\lambda}(1+C_t) \le \frac{\epsilon}{4 \overline{c}^u_{t,\lambda}}.$$

This gives $\widehat{\beta}_{i,n} \overline{c}^u_{t,\lambda} \le \epsilon^2/4$. On the other hand, we have

$$\int_{[-1/n, a_{t,\lambda}]} x \nu_{i,n}(dx) = \sum_{k=0}^{a_{t,\lambda} n+1} k \mu_{i,n}\{k\} - \mu_{i,n}[0, a_{t,\lambda} n + 1] \ge \epsilon - 1.$$

Consider now any $\eta > 0$ such that $1 + (1+\eta)(\epsilon - 1) \ge \epsilon/2$ and using Lemmas 4.2 and 4.3 let $n_{t,\lambda,\eta}$ such that $\epsilon_{i,n}(u_n(t_{i+1}^n, t, \lambda)) \in (-\eta, \eta)$ for all $n \ge n_{t,\lambda,\eta}$ and all $0 \le i < \gamma_n(t)$. Then

$$\widehat{\alpha}_{i,n} = 1 + \left(1 + \epsilon_{i,n}(u_n(t_{i+1}^n, t, \lambda))\right) \int_{[-1/n, a]} x \nu_{i,n}(dx) \ge 1 + (1+\eta)(\epsilon - 1) \ge \epsilon/2$$

and so

$$(\widehat{\alpha}_{i,n})^2 - \overline{c}^u_{t,\lambda}\widehat{\alpha}_{i,n}\widehat{\beta}_{i,n} = \widehat{\alpha}_{i,n}\left(\widehat{\alpha}_{i,n} - \overline{c}^u_{t,\lambda}\widehat{\beta}_{i,n}\right) \geq \frac{\epsilon}{2}\left(\frac{\epsilon}{2} - \frac{\epsilon}{4}\right) = \epsilon^2/8.$$

Then we can apply Lemma 5.2 to the sequence $w_i = u(t^n_i, t, \lambda)$ with $M = \overline{c}^u_{t,\lambda}$, $\epsilon = \epsilon^2/8$, $b_i = \widehat{\alpha}_{i,n}$ and $c_i = \widehat{\beta}_{i,n}$. We get at $i = \gamma_n(s)$

$$u_n(s,t,\lambda) \geq \left(\frac{1}{\lambda}\Gamma^{\gamma_n(t)-1}_{\gamma_n(s)} + \sum_{k=\gamma_n(s)}^{\gamma_n(t)-1}\Gamma^{k-1}_{\gamma_n(s)}\delta_k\right)^{-1} \tag{29}$$

where $\delta_k = c_k/b_k^2$, $\Gamma^{i-1}_i = 1$ and $\Gamma^i_j = \prod_{k=j}^i \gamma_k$ for $j \leq i$, with $\gamma_k = (1 + 8c_k M^2/\epsilon^2)/b_k$. Using $1 + x \leq e^x$ we obtain for $j \leq i$

$$\Gamma^i_j \leq \exp\left(-\sum_{k=j}^i \log\alpha_k\right)\exp\left(\sum_{k=j}^i \frac{8c_k M^2}{\epsilon^2}\right).$$

Since $c_k = \widehat{\beta}_{k,n} \leq 4\beta_{k,n}$ we have $\sum_{k=j}^i c_k \leq 4C_t$ for any $i < \gamma_n(t)$ and so

$$\Gamma^i_{\gamma_n(s)} \leq \exp\left(-\sum_{k=\gamma(s)}^i \log\widehat{\alpha}_{k,n}\right)\exp\left(\frac{32C_t M^2}{\epsilon^2}\right),\ i < \gamma_n(t).$$

Let $c_\epsilon > 0$ such that $\log(x) \geq c_\epsilon(x-1)$ for all $\epsilon/2 \leq x \leq 1$, so that $\log x \geq c_\epsilon(x-1)^-$ for all $x \geq \epsilon/2$, with $x^- = x$ if $x \leq 0$ and $x^- = 0$ if $x \geq 0$. Since $\widehat{\alpha}_{k,n} \geq \epsilon/2$ we obtain

$$\log(\widehat{\alpha}_{k,n}) \geq c_\epsilon\left(\widehat{\alpha}_{k,n} - 1\right)^- \geq c_\epsilon\left(-2\left(|\alpha_{i,n}| + a^{-1}_{t,\lambda}\beta_{i,n} + \mu_{i,n}\{0\}/n^2\right)\right)^-.$$

Hence for any $j \leq i < \gamma_n(t)$ we have $\sum_{k=j}^i \log(\widehat{\alpha}_{k,n}) \geq -2c_\epsilon(2 + a^{-1}_{t,\lambda})C_t$ and so for any $k < \gamma_n(t)$ we obtain

$$\Gamma^k_{\gamma_n(s)} \leq \exp\left(2c_\epsilon(2 + a^{-1}_{t,\lambda})C_t\right)\exp\left(32C_t M^2/\epsilon^2\right).$$

Plugging this into (29) we obtain

$$u_n(s,t,\lambda) \geq \exp\left(-2c_\epsilon(2 + a^{-1}_{t,\lambda})C_t - 32C_t M^2/\epsilon^2\right)\left(\frac{1}{\lambda} + \sum_{k=\gamma_n(s)}^{\gamma_n(t)-1}\delta_k\right)^{-1}.$$

Using $\widehat{\alpha}_{k,n} \geq \epsilon/2$ and $\sum\widehat{\beta}_{k,n} \leq 4C_t$ we obtain $\sum_{k=\gamma_n(s)}^{\gamma_n(t)-1}\delta_k \leq 4\epsilon^{-2}4C_t$ which finally proves that

$$\liminf_{n\to+\infty}\left(\inf_{0\leq s\leq t} u_n(s,t,\lambda)\right) > 0.$$

In view of Lemma 4.5 this proves (4), hence the result. □

5.2. **Proof of Corollary 2.5.** Assume now that (A1) and (A2) hold. Assume first that (5) holds, we must prove that $\liminf_{n\to\infty} u_n(s,t,\lambda) = 0$ for every $s < \wp(t)$. Fix until the end of the proof $t, \lambda \geq 0$, and imagine for a moment that we would know that

$$\lim_{\lambda\to 0}\left(\sup_{n\geq 1, v\in[s,t]} u_n(s,v,\lambda)\right) = 0 \tag{30}$$

for every $s \leq t$. Then, for every $s < \wp(t)$, Lemma 4.5 guarantees the existence of sequences $(n(k))$ and (v_k) such that $v_k \in [s,t]$ for each $k \geq 1$, $n(k) \to \infty$ and $u_{n(k)}(v_k, t, \lambda) \to 0$ as $k \to +\infty$. Then, the composition rule (1) shows that for every $k \geq 1$,

$$u_{n(k)}(s,t,\lambda) = u_{n(k)}\left(s, v_k, u_{n(k)}(v_k,t,\lambda)\right) \leq \sup_{n\geq 1, v\in[s,t]} u_n\left(s, v, u_{n(k)}(v_k,t,\lambda)\right).$$

Since $u_{n(k)}(v_k, t, \lambda) \to 0$ as $k \to +\infty$, (30) implies that $u_{n(k)}(s, t, \lambda) \to 0$ which shows that $\liminf_{n\to\infty} u_n(s, t, \lambda) = 0$. We now show (30) using (5). So in addition to t and λ, fix until further notice $s \le t$. Then for any $n \ge 1$, $v \in [s, t]$ and $A > 0$ we first write

$$1 - e^{-u_n(s,v,\lambda)} = 1 - \mathbb{E}\left(\exp(-\lambda X_n(v)) \mid X_n(s) = 1\right) \le 1 - e^{-\lambda A} + \mathbb{P}(X_n(v) \ge A \mid X_n(s) = 1)$$

which gives $1 - e^{-u_n(s,v,\lambda)} \le 1 - e^{-\lambda A} + f_{s,t}(A)$ with

$$f_{s,t}(A) = \sup_{n\ge1, v\in[s,t]} \mathbb{P}(X_n(v) \ge A \mid X_n(s) = 1).$$

In particular, $u_n(s, v, \lambda) \le -\log\left(e^{-\lambda A} - f_{s,t}(A)\right)$ which entails

$$\sup_{n\ge1, v\in[s,t]} u_n(s, v, \lambda) \le -\log\left(e^{-\lambda A} - f_{s,t}(A)\right).$$

This proves (30), letting first $\lambda \to 0$ and then $A \to +\infty$, since $f_{s,t}(A) \to 0$ by assumption. We now assume that

$$\sup_{n\ge1}\left(\int_{[-1/n,\infty)\times(0,t]} |x| \nu_n(dx\,dy)\right) < +\infty$$

and we prove that this implies (5). Fix $s \le t$: by Lemma 4.3 there exists a finite constant C_t such that $u_n(s, v, \lambda) \le C_t$ for all $v \in [s, t]$ and $\lambda \le 1$. Further, by Lemma 4.2 there exists n_t such that $|\epsilon_{i,n}(y)| \le 1$ for any $n \ge n_t$, $y \le C_t$ and $i \le \gamma_n(t)$. Finally, invoking Lemma 4.1 and using $1 - \exp(-\lambda x) \le \lambda |x|$ for $x \in \mathbb{R}$ and $\lambda \ge 0$, we get

$$u_n(s, t, \lambda) \le \lambda + 2 \sum_{i=\gamma_n(s)+1}^{\gamma_n(t)} u_n(t_i^n, t, \lambda) \int_{[-1/n,\infty)} |x| \nu_{i-1,n}(dx) = \lambda + 2 \int_{(s,t]} u_n(y, t, \lambda) \widetilde{\nu}_n(dy)$$

with $\widetilde{\nu}_n(dy) = \int_{[-1/n,\infty)} |x| \nu_n(dx\,dy)$. Then $\widetilde{\nu}_n$ is a σ-finite measure and Lemma 4.6 implies that

$$u_n(s, t, \lambda) \le \lambda + \lambda \widetilde{\nu}_n(s, t] e^{\widetilde{\nu}_n(s,t]}.$$

This shows that

$$\sup_{n\ge1, v\in[s,t]} u_n(s, v, \lambda) \le \lambda\left(1 + \widetilde{\nu}_n(s, t] e^{\widetilde{\nu}_n(s,t]}\right) \underset{\lambda\to0}{\longrightarrow} 0.$$

To see that this implies (5), we write for any $A \ge 1$

$$\mathbb{P}\left(X_n(v) \le A \mid X_n(s) = 1\right) \le \mathbb{P}\left(1 - e^{-X_n(v)/A} \ge 1 - 1/e \mid X_n(s) = 1\right) \le \frac{1 - e^{-u_n(s,v,1/A)}}{1 - 1/e}.$$

APPENDIX A. PROOF OF LEMMA 4.8

A.1. **Preliminary lemmas.** This appendix is devoted to the proof of Lemma 4.8. We begin with two preliminary lemmas. Recall that $\mu = |\alpha| + \beta$.

Lemma A.1. *For any $\varepsilon > 0$ and $0 \le s < t$, there exists a partition of the interval $(s, t]$ as*

$$(s, t] = \left(\bigcup_{j=1}^{J} (a_j, b_j]\right) \cup \left(\bigcup_{k=1}^{K} (a'_k, b'_k]\right)$$

such that $\mu(a_j, b_j] \le \varepsilon$ for each $1 \le j \le J$, $\{b'_k, 1 \le k \le K\} = (s, t] \cap \{u \ge 0 : \mu\{u\} \ge \varepsilon\}$ and $\mu(a'_k, b'_k) \le \varepsilon/K$ for each $1 \le k \le K$.

Proof. As $\mu(0, t]$ is finite, we can define K and the (b'_k) such that $\{b'_k, 1 \le k \le K\} = (s, t] \cap \{v \ge 0 : \mu\{v\} \ge \varepsilon\}$. By left-continuity, for each b'_k one can choose $b'_{k-1} < a'_k < b'_k$ such that $\mu(a'_k, b'_k) \le \varepsilon$. The result follows since one can partition $(b'_k, a'_{k+1}]$ into (π_ℓ) such that $\mu(\pi_\ell, \pi_{\ell+1}] \le \varepsilon$ since by choice of b'_k and a_k, $(b'_k, a'_{k+1}]$ does not contain any atom of μ_n of size ε or larger. $\square$

Lemma A.2. *For any $t \geq 0$,*

$$\lim_{d\to 0}\limsup_{n\to+\infty}\int_{[-1/n,d)\times(0,t]}\frac{|x|^3}{1+x^2}\nu_n(dx\,dy)=\lim_{d\to 0}\int_{(0,d)\times(0,t]}\frac{x^3}{1+x^2}\nu(dx\,dy)=0.$$

Proof. We have

$$\begin{aligned}\int_{[-1/n,d)\times(0,t]}\frac{|x|^3}{1+x^2}\nu_n(dx\,dy)&\leq\left(\frac{1}{n}+d\right)\int_{[-1/n,d)\times(0,t]}\frac{x^2}{1+x^2}\nu_n(dx\,dy)\\&\leq\left(\frac{1}{n}+d\right)\int_{[-1/n,\infty)\times(0,t]}\frac{x^2}{1+x^2}\nu_n(dx\,dy)\end{aligned}$$

and this last term is equal to $2(1/n+d)\beta_n(t)$. Since $\beta_n(t)\to\beta(t)$ we get

$$\limsup_{n\to+\infty}\int_{[-1/n,d)\times(0,t]}\frac{|x|^3}{1+x^2}\nu_n(dx\,dy)\leq 2d\beta(t)$$

and since this last bound goes to 0 as $d\to 0$ this proves the first part of the lemma. The result for ν follows along the same lines, using (27). □

A.2. **Proof of Lemma 4.8.** In the rest of the proof fix $t,\lambda>0$, $\wp(t)<s\leq t$, (ℓ_n) a sequence converging to λ and note $u_n(y)=u_n(y,t,\ell_n)$. With these notation, we have

$$R_n(y)=\left|\Psi_n(u_n)((y,t])-\Psi(u_n)((y,t])\right|,\ 0\leq y\leq t.$$

Let $\ell=\inf_{n\geq 1}\ell_n$ and $L=\sup_{n\geq 1}\ell_n$ and assume without loss of generality, since $\ell_n\to\lambda>0$, that $\ell>0$. We first show that $R_n(s)\to 0$, the fact that $\sup\{R_n(y): s\leq y\leq t, n\geq 1\}$ is proved in Section A.2.3. From the definitions (14) and (16) of Ψ and Ψ_n one can write

$$|\Psi_n(u_n)((s,t])-\Psi(u_n)((s,t])|\leq B_n^\alpha+B_n^\beta+B_n^\nu+B_n^\epsilon$$

with

$$B_n^\alpha=\left|\int_{(s,t]}u_n\,d\alpha_n-\int_{(s,t]}u_n\,d\alpha\right|,\quad B_n^\beta=\left|\int_{(s,t]}u_n^2\,d\beta_n-\int_{(s,t]}u_n^2\,d\beta\right|,$$

$$B_n^\nu=\left|\int_{[-1/n,\infty)\times(s,t]}h(x,u_n(y))\nu_n(dx\,dy)-\int_{(0,\infty)\times(s,t]}h(x,u_n(y))\nu(dx\,dy)\right|$$

and

$$B_n^\epsilon=\sum_{i=\gamma_n(s)+1}^{\gamma_n(t)}\left|\epsilon_{i-1,n}(u_n(t_i^n))\right|\left|\int\left(1-e^{-xu_n(t_i^n)}\right)\nu_{i-1,n}(dx)\right|.$$

We will show that each sequence (B_n^α), (B_n^β), (B_n^ν) and (B_n^ϵ) goes to 0 as n goes to infinity. By (20) and by definition of the constants $\overline{c}^\epsilon_{n,t}$, $\overline{c}^u_{t,L}$ and c_1, one can derive similarly as in the proof of Lemma 4.4

$$B_n^\epsilon\leq\overline{c}^\epsilon_{n,t}(\overline{c}^u_{t,\ell_n})c_1(\overline{c}^u_{t,\ell_n})\mu_n(t)\leq\overline{c}^\epsilon_{n,t}(\overline{c}^u_{t,L})c_1(\overline{c}^u_{t,L})\mu_n(t)$$

where the last inequality follows from the fact that $\ell_n\leq L$ and that the functions $\overline{c}^\epsilon_{n,t}(C)$ and $\overline{c}^u_{t,y}$ are increasing in C and y, respectively. From now, we will use such monotonicity properties without further comment. This last upper bound is seen to go 0, invoking (25) and Lemmas 4.2 and 4.3. Thus the sequence (B_n^ϵ) goes to 0 and we have to control the three other sequences (B_n^α), (B_n^β) and (B_n^ν). We control the two first sequences in Section A.2.1 and the last one in Section A.2.2

A.2.1. *Control of the sequences* (B_n^α) *and* (B_n^β). We treat in detail the convergence of (B_n^α) to 0. For (B_n^β), one essentially needs to replace α by β and u_n by u_n^2, we mention along the way what modifications need to be done.

Fix $\varepsilon > 0$ and consider the partition $((a_j, b_j], 1 \le j \le J)$ and $((a'_k, b'_k], 1 \le k \le K)$ of $(s,t]$ provided by Lemma A.1. Note that the partition depends on s, t and ε but not on n. We can write $B_n^\alpha \le \sum_{j=1}^J B_{n,j}^{\alpha,1} + \sum_{k=1}^K (B_{n,k}^{\alpha,2} + B_{n,k}^{\alpha,3})$ with

$$B_{n,j}^{\alpha,1} = \left| \int_{(a_j,b_j]} u_n d\alpha_n - \int_{(a_j,b_j]} u_n d\alpha \right|, \quad B_{n,k}^{\alpha,2} = \int_{(a'_k,b'_k)} u_n d|\alpha_n| + \int_{(a'_k,b'_k)} u_n d|\alpha|$$

and $B_{n,k}^{\alpha,3} = u_n(b'_k)\left|\alpha_{\gamma_n(b'_k),n} - \alpha\{b'_k\}\right|$. For $B_{n,j}^{\alpha,1}$ we have

$$\begin{aligned} B_{n,j}^{\alpha,1} \le \int_{(a_j,b_j]} \left|u_n(y) - u_n(b_j)\right| |\alpha_n|(dy) + \int_{(a_j,b_j]} \left|u_n(y) - u_n(b_j)\right| |\alpha|(dy) \\ + u_n(b_j)\left|\alpha_n(a_j,b_j] - \alpha(a_j,b_j]\right|. \end{aligned}$$

By Lemma 4.4, $\left|u_n(y) - u_n(b_j)\right| \le \Delta_{t,L}^u \mu_n(a_j,b_j]$ for all $y \in (a_j,b_j]$ and so, using also $u_n(b_j) \le \overline{c}_{t,L}^u$, we get

$$B_{n,j}^{\alpha,1} \le \Delta_{t,L}^u \mu_n(a_j,b_j]\big(|\alpha_n|(a_j,b_j] + |\alpha|(a_j,b_j]\big) + \overline{c}_{t,L}^u \left|\alpha_n(a_j,b_j] - \alpha(a_j,b_j]\right|.$$

For B_n^β one needs to use

$$\left|u_n(y)^2 - u_n(b_j)^2\right| = \left|u_n(y) - u_n(b_j)\right| \big(u_n(y) + u_n(b_j)\big) \le 2\overline{c}_{t,L}^u \Delta_{t,L}^u \mu_n(a_j,b_j]$$

, which leads to a similar upper bound. Since the partition does not depend on n, we have $\alpha_n(a_j,b_j] \to \alpha(a_j,b_j]$ and $\mu_n(a_j,b_j] \to \mu(a_j,b_j]$ by assumption (A1), so that summing over $j = 1, \dots, J$, letting n go to infinity and using $|\alpha|(A) \le \mu(A)$ gives

$$\limsup_{n\to+\infty} \sum_{j=1}^J B_{n,j}^{\alpha,1} \le 2\Delta_{t,L}^u \sum_{j=1}^J \big(\mu(a_j,b_j]\big)^2 \le 2\varepsilon \Delta_{t,L}^u \mu(s,t], \tag{31}$$

using also $\mu(a_j,b_j] \le \varepsilon$, which holds by choice of the partition, to derive the second inequality. To upper bound $B_{n,k}^{\alpha,2}$ we write $B_{n,k}^{\alpha,2} \le \overline{c}_{t,L}^u \left(|\alpha_n|(a'_k,b'_k) + |\alpha|(a'_k,b'_k)\right)$ which leads, using $\mu(a'_k,b'_k) \le \varepsilon/K$, to

$$\limsup_{n\to+\infty} \sum_{k=1}^K B_{n,k}^{\alpha,2} \le 2\overline{c}_{t,L}^u \sum_{k=1}^K \mu(a'_k,b'_k) \le 2\varepsilon \overline{c}_{t,L}^u. \tag{32}$$

For $B_{n,k}^{\beta,2}$ one can use $B_{n,k}^{\beta,2} \le (\overline{c}_{t,L}^u)^2 \left(\beta_n(a'_k,b'_k) + \beta(a'_k,b'_k)\right)$ to obtain a similar upper bound.

Finally for $B_{n,k}^{\alpha,3}$ one has $B_{n,k}^{\alpha,3} \le \overline{c}_{t,L}^u |\alpha_{\gamma_n(b'_k),n} - \alpha(\{b'_k\})|$ which goes to 0 by assumption (A2). One can similarly write $B_{n,k}^{\beta,3} \le (\overline{c}_{t,L}^u)^2 |\beta_{\gamma_n(b'_k),n} - \beta\{b'_k\}|$ for $B_{n,k}^{\beta,3}$. Since K does not depend on n this gives $\sum_{k=1}^K B_{n,k}^{\alpha,3} \to 0$ and so (31) and (32) give

$$\limsup_{n\to+\infty} B_n^\alpha \le 2\varepsilon \left(\Delta_{t,L}^u \mu(s,t] + \overline{c}_{t,L}^u\right).$$

Since ε was arbitrary, letting $\varepsilon \to 0$ gives the result.

A.2.2. *Control of the sequence* (B_n^ν). For $T \geq 0$ we define the constant

$$c_4(T) = \sup\left\{\left|\frac{h(x,y)}{x^3/(1+x^2)}\right| : x \geq -1, 0 \leq y \leq T\right\} \tag{33}$$

which, starting from (19), can be seen to be finite. For $d > 0$ we write

$$B_n^\nu \leq \widetilde{B}_n^\nu + \hat{B}_n^\nu + \check{B}_n^\nu \tag{34}$$

with

$$\widetilde{B}_n^\nu = \left|\int_{[d,\infty)\times(s,t]} h(x,u_n(y))\nu_n(dx\,dy) - \int_{[d,\infty)\times(s,t]} h(x,u_n(y))\nu(dx\,dy)\right|.$$

$$\hat{B}_n^\nu = \int_{[-1/n,d)\times(s,t]} |h(x,u_n(y))|\nu_n(dx\,dy), \quad \check{B}_n^\nu = \int_{(0,d)\times(s,t]} |h(x,u_n(y))|\nu(dx\,dy).$$

Note that $\widetilde{B}_n^\nu$ depends on d but, similarly as t or λ, we do not reflect this in the notation because d will be fixed once and for all shortly. Bounding the two last terms thanks to (33), we have

$$B_n^\nu \leq \widetilde{B}_n^\nu + c_4(\overline{c}_{t,L}^u)\left(\int_{(0,d)\times(0,t]} \frac{x^3}{1+x^2}\nu(dx\,dy) + \int_{[-1/n,d)\times(s,t]} \frac{|x|^3}{1+x^2}\nu_n(dx\,dy)\right).$$

Letting first $n \to +\infty$ and then $d \to 0$, we obtain by Lemma A.2

$$\limsup_{n\to+\infty} B_n^\nu \leq \lim_{d\to 0}\limsup_{n\to+\infty} \widetilde{B}_n^\nu.$$

Hence to prove $B_n^\nu \to 0$ we only have to show that $\widetilde{B}_n^\nu \to 0$ for every $d > 0$. So in the rest of this step we fix an arbitrary $d > 0$ and show that $\widetilde{B}_n^\nu \to 0$. Fix $\varepsilon > 0$ and consider the partition $((a_j, b_j], 1 \leq j \leq J)$ and $((a'_k, b'_k], 1 \leq k \leq K)$ of $(s,t]$ given by Lemma A.1, which does not depend on n. Then we can write $\widetilde{B}_n^\nu \leq \sum_{j=1}^J \widetilde{B}_{n,j}^{\nu,1} + \sum_{k=1}^K (\widetilde{B}_{n,k}^{\nu,2} + \widetilde{B}_{n,k}^{\nu,3})$ with

$$\widetilde{B}_{n,j}^{\nu,1} = \left|\int_{[d,\infty)\times(a_j,b_j]} h(x,u_n(y))\nu_n(dx\,dy) - \int_{[d,\infty)\times(a_j,b_j]} h(x,u_n(y))\nu(dx\,dy)\right|,$$

$$\widetilde{B}_{n,k}^{\nu,2} = \int_{[d,\infty)\times(a_k,b'_k)} |h(x,u_n(y))|\nu_n(dx\,dy) + \int_{[d,\infty)\times(a_k,b'_k)} |h(x,u_n(y))|\nu(dx\,dy)$$

and

$$\widetilde{B}_{n,k}^{\nu,3} = \left|\int_{[d,\infty)} h(x,u_n(b'_k))\nu_{\gamma_n(b'_k),n}(dx) - \int_{[d,\infty)\times\{b'_k\}} h(x,u_n(b'_k))\nu(dx\,dy)\right|.$$

Further we write $\widetilde{B}_{n,j}^{\nu,1} \leq \widetilde{B}_{n,j}^{\nu,4} + \widetilde{B}_{n,j}^{\nu,5}$ with

$$\widetilde{B}_{n,j}^{\nu,4} = \int_{[d,\infty)\times(a_j,b_j]} \left|h(x,u_n(y)) - h(x,u_n(b_j))\right|\nu_n(dx\,dy) + \int_{[d,\infty)\times(a_j,b_j]} \left|h(x,u_n(y)) - h(x,u_n(b_j))\right|\nu(dx\,dy)$$

and

$$\widetilde{B}_{n,j}^{\nu,5} = \left|\int_{[d,\infty)\times(a_j,b_j]} h(x,u_n(b_j))\nu_n(dx\,dy) - \int_{[d,\infty)\times(a_j,b_j]} h(x,u_n(b_j))\nu(dx\,dy)\right|.$$

We derive, in order, upper bounds for $\widetilde{B}_{n,k}^{\nu,2}$, $\widetilde{B}_{n,j}^{\nu,4}$, $\widetilde{B}_{n,k}^{\nu,3}$ and finally $\widetilde{B}_{n,j}^{\nu,5}$.

To control $\widetilde{B}^{\nu,2}_{n,k}$ we introduce the constant

$$c_5(T) = \sup\left\{\frac{|h(x,y)|}{x^2/(1+x^2)} : 0 \le y \le T, x \ge 0\right\}$$

which can be seen to be finite, starting from instance from (19). Thus

$$\begin{aligned}\widetilde{B}^{\nu,2}_{n,k} &= \int_{[d,\infty)\times(a'_k,b'_k)} |h(x,u_n(y))|\nu_n(dx\,dy) + \int_{[d,\infty)\times(a_k,b'_k)} |h(x,u_n(y))|\nu(dx\,dy)\\ &\le c_5(\overline{c}^u_{t,L})\left(\int_{[d,\infty)\times(a'_k,b'_k)} \frac{x^2}{1+x^2}\nu_n(dx\,dy) + \int_{[d,\infty)\times(a_k,b'_k)} \frac{x^2}{1+x^2}\nu(dx\,dy)\right)\\ &\le 2c_5(\overline{c}^u_{t,L})\left(\beta_n(a'_k,b'_k)+\beta(a'_k,b'_k)\right)\end{aligned}$$

using (27) for the last inequality. Using $\beta_n(a'_k,b'_k) \to \beta(a'_k,b'_k) \le \mu(a'_k,b'_k) \le \varepsilon/K$, this leads to

$$\limsup_{n\to+\infty} \sum_{k=1}^K \widetilde{B}^{\nu,2}_{n,k} \le 4\varepsilon c_5(\overline{c}^u_{t,L}). \tag{35}$$

To derive an upper bound on $\widetilde{B}^{\nu,4}_{n,j}$, we use the constant $c_2(\eta,T)$ defined in (26). Since $0 < \underline{c}^u_{s,t,\ell} \le u_n(y) \le \overline{c}^u_{t,L}$ for $n \ge N_{s,t,\ell}$ and $a_j < y \le b_j$, we have for such n

$$\begin{aligned}&\int_{[d,\infty)\times(a_j,b_j]} \left|h(x,u_n(y)) - h(x,u_n(b_j))\right| \nu_n(dx\,dy)\\ &\qquad\le c_2\left(\underline{c}^u_{s,t,\ell},\overline{c}^u_{t,L}\right)\int_{[d,\infty)\times(a_j,b_j]} \left|u_n(y)-u_n(b_j)\right| \frac{x^2}{1+x^2}\nu_n(dx\,dy).\end{aligned}$$

Since $|u_n(y)-u_n(b_j)| \le \Delta^u_{t,L}\,\mu_n(a_j,b_j]$ for $a_j < y \le b_j$ by Lemma 4.4, we obtain

$$\begin{aligned}&\int_{[d,\infty)\times(a_j,b_j]} \left|h(x,u_n(y)) - h(x,u_n(b_j))\right| \nu_n(dx\,dy)\\ &\qquad\le c_2\left(\underline{c}^u_{s,t,\ell},\overline{c}^u_{t,L}\right)\Delta^u_{t,L}\,\mu_n(a_j,b_j]\int_{[d,\infty)\times(a_j,b_j]} \frac{x^2}{1+x^2}\nu_n(dx\,dy).\end{aligned}$$

Since

$$\int_{[d,\infty)\times(a_j,b_j]} \frac{x^2}{1+x^2}\nu_n(dx\,dy) \le \int_{[-1/n,\infty)\times(a_j,b_j]} \frac{x^2}{1+x^2}\nu_n(dx\,dy) = 2\beta_n(a_j,b_j]$$

and $\beta_n(a_j,b_j] \le \mu_n(a_j,b_j]$ we finally get

$$\int_{[d,\infty)\times(a_j,b_j]} \left|h(x,u_n(y)) - h(x,u_n(b_j))\right| \nu_n(dx\,dy) \le C_{s,t,\ell,L}(\mu_n(a_j,b_j])^2$$

with $C_{s,t,\ell,L} = 2c_2(\underline{c}^u_{s,t,\ell},\overline{c}^u_{t,L})\Delta^u_{t,L}$. The exact same reasoning with ν instead of ν_n, using the inequality (27) instead of the equality

$$\int_{[-1/n,\infty)\times(a_j,b_j]} \frac{x^2}{1+x^2}\nu_n(dx\,dy) = 2\beta_n(a_j,b_j],$$

leads to

$$\widetilde{B}^{\nu,4}_{n,j} \le C_{s,t,\ell,L}\left[(\mu_n(a_j,b_j])^2 + (\mu(a_j,b_j])^2\right].$$

Hence (25) gives

$$\limsup_{n\to+\infty} \sum_{j=1}^J \widetilde{B}^{\nu,4}_{n,j} \le 2C_{s,t,\ell,L}\sum_{j=1}^J (\mu(a_j,b_j])^2 \le 2\varepsilon C_{s,t,\ell,L}\mu(s,t] \tag{36}$$

using $\mu(a_j, b_j] \le \varepsilon$ to get the second inequality.

The arguments to control $\widetilde{B}^{\nu,3}_{n,k}$ and $\widetilde{B}^{\nu,5}_{n,j}$ are very similar: we treat the case $\widetilde{B}^{\nu,5}_{n,j}$ in detail and mention necessary changes needed for $\widetilde{B}^{\nu,3}_{n,k}$. We need the constant c_6

$$c_6(T) = \sup_{\substack{0\le y\le T\\ 0\le x,x'}} \left| \frac{h(x,y)-h(x',y)}{x-x'} \right| \tag{37}$$

which is finite because

$$\frac{\partial h}{\partial x}(x,y) = ye^{-xy} + y\frac{x^2+xy-1}{(1+x^2)^2}$$

and so for $x, x' \ge 0$ and $0 \le y \le T$,

$$\left|\frac{\partial h}{\partial x}(x,y)\right| \le T + T\sup_{v\ge 0}\left(\frac{v^2+Tv+1}{(1+v^2)^2}\right).$$

Let $\pi_{n,j}$ be the signed measures defined for $A \in \mathscr{B}$ by

$$\pi_{n,j}(A) = \nu_n(A\times(a_j,b_j]) - \nu(A\times(a_j,b_j]).$$

For $\widetilde{B}^{\nu,3}_{n,k}$ one needs to consider the measure $\pi_{n,k}$ defined similarly but with $A\times\{b'_k\}$ instead of $A\times(a_j,b_j]$. With this notation we have

$$\widetilde{B}^{\nu,5}_{n,j} \le \sup_{0\le y\le \overline{c}^u_{t,L}} \left|\int_{[d,\infty)} h(x,y)\pi_{n,j}(dx)\right|.$$

Fix $Y,\eta>0$ and consider a subdivision $d=\tau_1<\cdots<\tau_N<\tau_{N+1}=\infty$ with the following three properties: (1) $\tau_{\ell+1}-\tau_\ell\le\eta$ for all $1\le\ell<N$; (2) $\tau_N = Y$; and (3) $\nu(\{\tau_\ell\}\times(a_j,b_j]) = 0$ for all $1\le\ell\le N$. For $\widetilde{B}^{\nu,3}_{n,k}$ the third condition should be $\nu(\{\tau_\ell\}\times\{b'_\ell\}) = 0$ for all $1\le\ell\le N$. Then for any $y\ge 0$,

$$\begin{aligned}\left|\int_{[d,\infty)} h(x,y)\pi_{n,j}(dx)\right| &\le \sum_{\ell=1}^{N-1}\int_{[\tau_\ell,\tau_{\ell+1})} |h(x,y)-h(\tau_\ell,y)||\pi_{n,j}|(dx)\\ &\quad + \int_{[Y,\infty)} |h(x,y)-h(Y,y)||\pi_{n,j}|(dx) + \sum_{\ell=1}^{N}|h(\tau_\ell,y)|\left|\pi_{n,j}([\tau_\ell,\tau_{\ell+1}))\right|.\end{aligned}$$

By choice of the partition (τ_ℓ) and by definition (37) of c_6, we have for any $y\le\overline{c}^u_{t,L}$

$$\begin{aligned}\sum_{\ell=1}^{N-1}\int_{[\tau_\ell,\tau_{\ell+1})} |h(x,y)-h(\tau_\ell,y)||\pi_{n,j}|(dx) &\le c_6(\overline{c}^u_{t,L})\sum_{\ell=1}^{N-1}\int_{[\tau_\ell,\tau_{\ell+1})}|x-\tau_\ell||\pi_{n,j}|(dx)\\ &\le \eta c_6(\overline{c}^u_{t,L})|\pi_{n,j}|([d,\infty)).\end{aligned}$$

Thus introducing the constant

$$\overline{c}^h_{t,L} = \sup\left\{|h(x,y)| : x\ge 0, 0\le y\le \overline{c}^u_{t,L}\right\}$$

which in view of (19) can be seen to be finite, one gets for any $y\le\overline{c}^u_{t,L}$,

$$\begin{aligned}\left|\int_{[d,\infty)} h(x,y)\pi_{n,j}(dx)\right| &\le \eta c_6(\overline{c}^u_{t,L})|\pi_{n,j}|[d,\infty) + 2\overline{c}^h_{t,L}|\pi_{n,j}|[Y,\infty)\\ &\qquad + \overline{c}^h_{t,L}\sum_{\ell=1}^{N}\left|\pi_{n,j}[\tau_\ell,\tau_{\ell+1})\right|.\end{aligned}$$

Since no (τ_ℓ) is an atom of the measure $\int_{\cdot\times(a_j,b_j]}\nu(dx\,dy)$, it follows from Assumption (A1) that $\pi_{n,j}[\tau_\ell,\tau_{\ell+1})\to 0$ as n goes to infinity for each ℓ. Moreover, one has

$$|\pi_{n,j}(A)|\le \nu_n(A\times(a_j,b_j])+\nu(A\times(a_j,b_j])$$

and finally, for any $\eta>0$ we have, using also the fact that $\limsup_{n\to\infty}|\pi_{n,j}|([c,\infty))\le 2\nu([c,\infty)\times(a_j,b_j])$ for any $c\ge 0$,

$$\limsup_{n\to+\infty}\sup_{0\le y\le \overline{c}^u_{t,L}}\left|\int_{[d,\infty)}h(x,y)\pi_{n,j}(dx)\right|\le 2\eta c_6(\overline{c}^u_{t,L})\nu([d,\infty)\times(a_j,b_j]) + 4\overline{c}^h_{t,L}\nu([Y,\infty)\times(a_j,b_j]).$$

Thus letting $\eta\to 0$ and $Y\to+\infty$ finally shows that $\widetilde{B}^{\nu,5}_{n,j}\to 0$ for each $1\le j\le J$ and also $\widetilde{B}^{\nu,3}_{n,k}\to 0$ for each $1\le k\le K$. Hence combining (35) and (36) finally gives

$$\limsup_{n\to+\infty}\widetilde{B}^\nu_n\le\varepsilon\left[c_5(\overline{c}^u_{t,L})+2C_{s,t,\lambda}\mu(s,t]\right]$$

and since ε is arbitrary, letting $\varepsilon\to 0$ achieves to prove that $R_n(s)\to 0$.

A.2.3. *Boundedness of* $(R_n(y))$. We now complete the proof of the lemma by showing that $\sup\{R_n(y):0\le y\le t, n\ge 1\}$ is finite. We have $R_n(y)\le\left|\Psi_n(u_n)((y,t])\right|+\left|\Psi(u_n)((y,t])\right|$, so that it is enough to prove that

$$\sup\left\{\left|\Psi_n(u_n)((y,t])\right|:0\le y\le t, n\ge 1\right\}<+\infty \tag{38}$$

and similarly with Ψ instead of Ψ_n. Using (17) for the first equality and Lemma 4.4 for the second inequality, we get for any $0\le y\le t$

$$\left|\Psi_n(u_n)((y,t])\right|=\left|u_n(y)-u_n(t)\right|\le\Delta^u_{t,L}\,\mu_n(y,t]\le\Delta^u_{t,L}\sup_{n\ge 1}\mu_n(t)$$

so that (38) holds. On the other hand, starting from the definition (14) of Ψ we get

$$\begin{aligned}\left|\Psi(u_n)((y,t])\right| &\le \int_{(s,t]}|u_n|d|\alpha|+\int_{(s,t]}(u_n)^2d\beta+\int_{(0,\infty)\times(s,t]}|h(x,u_n(y))|\nu(dx\,dy)\\ &\le \overline{c}^u_{t,L}|\alpha|(t)+(\overline{c}^u_{t,L})^2\beta(t)+c_5(\overline{c}^u_{t,L})\int_{(0,\infty)\times(0,t]}\frac{x^2}{1+x^2}\nu(dx\,dy)\end{aligned}$$

which ends the proof of the lemma, since this upper bound is finite (invoking (27) for the finiteness of the integral term).

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